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THEORY OF SOLID AND BRACED ELASTIC ARCHES,

APPLIED TO ARCH BRIDGES AND ROOFS IN IRON, WOOD, CONCRETE OR OTHER MATERIAL. GRAPHICAL ANALYSIS.

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III.

ARCH FIXED AT THE ENDS AND HINGED AT THE CROWN.

72. Let the curve AaB, Fig. 12, represent the neutral line of an arch-rib, fixed in direction and position at A and B, and hinged, or free to turn, at a point D

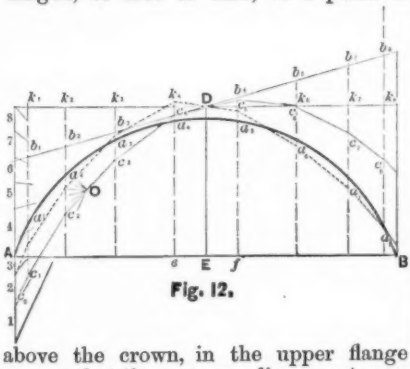


Fig. 12.

above the crown, in the upper flange say, so that the pressure line must pass through D.

Divide the arc into 8 equal parts say, and draw ordinates through the middle of each part at a_1, a_2, ... Lay off the loads 1, 2, . . . , acting at a_1, a_2, on the left vertical; assume a pole o and draw the equilibrium polygon cc. We have caused it to pass through D for convenience. If bb is a trial closing line, for the rib acting as a girder, the moments at a_1, a_2,.., are proportional to $\overline{b_1c_1}$,$\overline{b_2c_2}$,.,.

Now by arts. 17, 19 and 25 we must have,

$$\sum_{A}^{D}(bc.x)=\sum_{B}^{D}(\overline{bc}.x);$$

(D being the origin of co-ordinates, x horizontal, y vertical) since the vertical deflection of D for the two halves of the arch is the same, owing to the hinge.

The condition above shows that the closing line bb must have such a position, that if the ordinates b_1c_1, b_2c_2, . . . , b_7c_7, are taken as forces acting in position, their resultant must act through D; for the left member represents the sum of the moments of these supposed forces for the left half of the arch, which must equal a similar sum for the right half of the arch.

By aid of an equilibrium polygon or

otherwise, find the position of the resultant of the supposed forces b_1c_1, b_2c_2, .., and suppose it to pass through e, to the left of the verticle DE. Now lower slightly the left end of line $b_1 b_8$; find the resultant of the new forces $b_1 a_1$, etc., and suppose it to pass through f, to the right of E. If the left end of $b_1 b_8$ be now raised in about the ratio of $\overline{Ef}$ to $\overline{Ee}$ of the last change, its position will be very nearly correct. It may be tested, &c. as before. Suppose now, that $b_1 b_8$ is the correct position of this closing line. It is evident that it is the best to fix the line $b_1 b_8$ as nearly correct in the first instance as possible. A more precise method can be given, but the above is probably as direct as any.

73. The closing line kk of the arc AaB, regarded as an equilibrium polygon, must, of course, pass through D; and since the sum of the moments of k_1a_1, k_2a_2, . . ., to the left of D, must equal those to the right of D, as for the girder, the closing line must be horizontal, since the arch is symmetrical about the crown. The ordinates of the type ka are now proportional to M_a.

74. From the third condition, art. 17, which shows that the horizontal displacement of D for both halves of the arch is the same, though in opposite directions, we have

$$\sum_{A}^{D}(M_c - M_a)y = -\sum_{B}^{D}(M_c - M_a)y.$$

It is plain that if the left member is plus, that the sum in the right member must be minus, for the bending of one half of the arch is different in kind from that of the other half; hence we have added a minus sign to make the final result plus. From this equation we derive

$$\sum_{A}^{D} M_c y + \sum_{B}^{D} M_c y = \sum_{A}^{D} M_a y + \sum_{B}^{D} M_a y,$$

all ordinates being plus.

Now M_c and M_a are proportional to the ordinates of the type $\overline{bc}$, $\overline{ka}$, respectively; also at a_1, a_2, .., y has the values k_1a_1, k_2a_2, . . . , respectively, so that the preceding products are easily found graphically as in art. 7.

If the two members of the equation are not equal, we must change the lengths of the ordinates b_1c_1, b_2c_2, . . . to make them so, since the right member is constant. This involves an opposite alteration in the pole distance. Now lay off the altered ordinates from k_1, k_2, &c. downwards, to locate the true pressure curve, as shown by the dotted line.

On drawing through o a line parallel to b_1b_8, we divide the load line 1 .. 8 into the vertical components of the reactions. Draw through this point of division a horizontal equal to the new pole distance to locate the new pole. These last lines are not drawn to avoid confusing the figure. The pressure curve may be tested to see if ΣMy for both halves of the arch is the same. We have now the reactions in position, together with the pressure curve from which the strains can all be found.

75. *Temperature Strains.*—The moments caused by a real or virtual alteration of span, must cause the same to rise or fall at D for one-half of the arch as the other; so that k_1k_8, as previously determined from this condition, is the line along which the horizontal thrust H due to temperature, &c. acts. The moment at any point as a_2 is therefore $H.\overline{a_2k_2}$. We determine H exactly as in art. 55, noting that in the second equation of that article, that $\overline{ak}$ and y are equal for this case, the ordinates being measured from k_1k_8 downwards.

On measuring the ordinates of the type ka in Fig. 12 to scale, squaring and adding, we find the $\Sigma(\overline{ak}.y)$ of art. 55, whence the value of H may be found.

ARCH RIB FIXED AT ONE END, AND JOINTED AT THE OTHER END AND AT THE MIDDLE.

76. Let the arch, Fig. 12, be fixed at B and jointed at A and D. This is only a particular case of the preceding construction. Since there can be no bending moment at A or at D, the closing line k_1k_8 must pass through these points, so that its position is at once given. Suppose it drawn, and its intersections with the ordinates marked k_1, k_2, . . . , as in the figure; then we have only to determine the new pole distance for polygon c, with the resulting change in the length of ordinates bc. The closing line $b_1 b_8$ must also pass through D, and C_8 must coincide with it.

77. Now the horizontal displacement of D for both halves must be the same,

though in opposite directions from their abutments; hence the preceding conditions, that may be written,

$$\sum_{A}^{D}(\overline{bc}.y)+\sum_{B}^{D}(\overline{bc}.y)=\sum_{A}^{D}(\overline{ak}.y)+\sum_{B}^{D}(\overline{ak}.y)$$

apply, the ordinates $\overline{bc}$ and $\overline{ak}$ being all regarded as plus. (Prof. Eddy, in his treatment of this case, takes the differences of quantities, proportional to those in each member, in place of the sum as above. This is plainly an oversight, as we see from the connection with the preceding case). Proceeding now as before we locate the true pressure curve.

78. The thrust due to temperature must act along the closing line $k_1 k_2$, drawn through A and D. Call F the inclined force so acting, its horizontal component being H. The moment at any point a_2, is then H. $\overline{a_2 k_2}$, or of the type H.$\overline{ak}$; so that the principles of art. 55 apply, except that the moments either side of the center not being the same, we must find the sum of the products of the type $(\overline{ak}.y)$ for the entire arch, $\overline{ak}$ being of different signs for the two halves of the arch.

$$\therefore\ \mathrm{H}\sum_{B}^{A}(\overline{ak}.y)=\frac{h\mathrm{EI}}{s}$$

The ordinates y are still reckoned from the horizontal through D, one of the hinged points, vertically downwards. We find H from the above equation, as illustrated in art. 56, from whence the strains due to the real or virtual change of span h are readily found.

ARCH WITH THREE HINGES.

79. In Fig. 13 we have represented one of the best forms of arch for short spans; for in consequence of its being free to turn at three points A, c and B, the upper chord being cut at d, and the

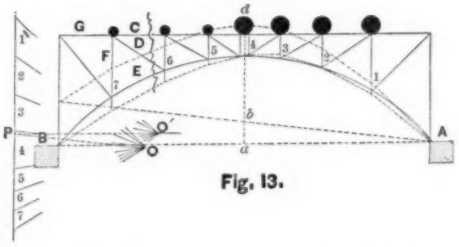


Fig. 13.

arch proper at c, there are no strains due to change of span, temperature and elastic shortening of the arch. Let us suppose that the two halves of the arch bear at c only. Suppose the right half of the arch to be loaded at the apices with the weights 1, 2, 3 and 4, shown on the left, due to dead and live load; the left half with the weights 5, 6, 7, due to dead load only. With an assumed pole o, draw the equilibrium polygon shown by the dotted line passing above d. Now since there can be no bending moments at a, c and B, the actual pressure curve must pass through these points. Therefore draw a line from o parallel to Ab, the closing line of the trial curve, to intersection with the load line 17, and from this point draw a horizontal, of a length equal to the old pole distance multiplied by the ratio of the ordinates $\overline{bd}$ and $\overline{ac}$, corresponding to the two curves at the center. This fixes the new pole o', from whence the true curve, passing through A, c and B, can be drawn.

This simple construction has been used by Prof. Eddy (see Researches in Graphical Statics) for passing a curve of pressures through three given points of a stone arch, as well as for the case above.

80. The pressure curve passes near the curved member from A to c, then keeps below the arc to B, for the loads assumed.

For a single weight, as that over apex 2, the pressure curve consists of two straight lines; one drawn from B, through c, to intersection with the vertical through 2, the other drawn from this last point to A. The strains

due to each weight may be found from its pressure curve, and tabulated; from whence the maximum strains that any member can ever be subjected to, from the most hurtful distribution of the load, can be ascertained.

METHOD OF FINDING THE STRAINS IN THE MEMBERS OF ANY ARCH.

81. The above figure will answer, by way of illustration, for the method to be pursued in any arch. Thus, suppose for any arch, fixed at the ends, &c., or otherwise shaped like Fig. 13, that we desire to know the stresses sustained by pieces, C, D and E, the resultant R=ray 0'67, acting along the side of the equilibrium polygon included between the verticals at 6 and 7.

Conceive a section, as shown by the wavy line, cutting C, D and E. Suppose, now, the arch to the right of the section removed, and its effect upon the left part replaced by forces acting *opposed to the resistances* in pieces C, D and E.

The part left of the section is still in equilibrium. Now, R, acting to the right, is the resultant, in position, of the reaction and loads *left of the section;* hence it must be in equilibrium with the *forces* applied to the cut pieces, C, D and E, which forces we may denote by the same letters as the pieces to which they correspond.

Therefore, the moment of R about any point, must equal the sum of the moments of C, D and E. Thus, take 6 as a center of moments; the moments of E and D are zero, hence the moment of C about 6 equals the moment of R, from whence C can be found.

Again, take apex DG as the center of moments, we have the moment of E equal to the moment of R about DG, from whence E follows.

We see in this case that the pieces C and E are in tension and compression respectively, since the forces C and E must act from and towards the cut pieces respectively to cause equilibrium with R.

It may be observed that when R is on the other side of the apex taken as the center of moments, that the strains caused are of an opposite character.

The strain in D can be found by taking moments about 7. The moment of R must equal the sum of the moments of C and D, etc.

Similarly, if we suppose E, F and G cut, and the part of the arch right of the section removed, including the weight at GD, supply forces E, F and G, opposed to the resistances in E, F, G; we have these forces in equilibrium with the resultant of the external forces to the left of the section, which is now simply the reaction at B (ray 0'7 P). With 7 as a center of moments we find G; and with the apex at the left end of G, as a center of moments, knowing E, we can find F; and so we proceed through the arch.

The well known Maxwell method of diagram may also be employed in this case, as illustrated by Du Bois in his Graphical Statics.

82. When the flanges of an arch are parallel, or nearly so, as usually happens, their strains may be determined as above; but the strains in the arch members are now very easily found by decom-

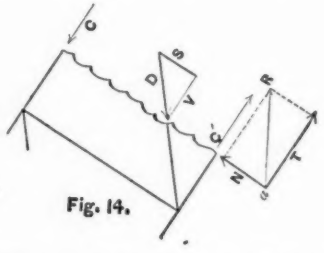


Fig. 14.

posing R, for the section taken, into components, N and T, normal and parallel to the arch at the section. This normal component, multiplied by the secant of its inclination to the diagonal cut, gives the strain in the latter. Thus in Fig. 14, let R, acting through a, be the resultant of all the forces to the left of the section, which must therefore be in equilibrium with the forces C, D, C', that are opposed to the resistances of the cut pieces. Now by mechanics, the algebraic sum of the tangential components are zero; *i.e.*, $C+V-C'-T=o$; also the sum of the normal components; *i.e.*, $S-N=o$, whence

$$S=N.$$

On multiplying N by the secant of the inclination of the diagonal to the normal, or laying off S to scale and drawing V parallel to T, we find the strain D in the diagonal.

83. The method of art. 11 applies here

also; both methods applying to the solid arch with a continuous web.

The proper resultant, acting at the supposed section, must be carefully noted; thus in Fig. 13, when E, F and G are the cut pieces, the resultant left of the section is the reaction acting through B, and not the resultant, 0·67 corresponding to the next section through E, D and C, since the weight at CG is then included.

Similarly in Fig. 11, for the case of the arch fixed at one end, the resultant for a section between A and a_1, *not* including the load at a_1, is the reaction passing through O; the resultant for any section between a_1 and a_2 is say 0″12, acting along $e_1 e_2$, and so on.

It is seen from Fig. 14, that when N acts upwards, that D is compression or tension, according as the top of the diagonal leans to the right or left of the normal; the reverse when N acts downwards.

84. The position of the live load that gives maximum stresses varies for each piece; so that it is only by computing the stresses caused by each single weight, and combining the resulting positive and negative strains, that the maximum of either kind may be found. As some of the positions of the live load may be rarely experienced (see "Maximum Stresses in Framed Bridges," by the writer, for similar facts in connection with simple girders), it is, of course, an open question, whether the effect of such unusual positions of the load may not be included under some per-centage added to usual strains. In all cases, however, it is correct practice to ascertain these max. strains, in order that the piece may be designed, so that for no loading, the limit of elasticity will be passed.

UNSYMMETRICAL ARCHES.

85. The preceding principles are applicable to unsymmetrical arches, as where one end of the arch is higher than the other, etc.; only the closing line of the arch regarded as an equilibrium polygon will not generally be horizontal, but must be drawn, as that in the polygon *c* due to the loading, to satisfy the proper conditions.

BRACED ARCHES.

86. The theory previously given is nearly correct for the *solid arch*, having a thin plate web (whose effects in causing a flexure is included in the term I), notwithstanding we have neglected the influence of the shearing or normal component N, in the formulæ of art 9, *et seq.*

If there are variations in E and I, for the different portions of the arch, we have seen that such variations are easily included in the graphical treatment, however difficult this may become in the strictly analytical solution. As a rule, however, the flanges, either in the solid or braced arch, are varied but little in size, except for very large spans, so that E as well as I is generally constant. But is the theory applicable to a *braced arch*, or one composed of two flanges with diagonal bracing? Since we cannot include this bracing, ordinarily, in the term I, its influence in causing bending deflection is neglected. In fact the strains on the diagonals are caused entirely by the normal components which are disregarded. The question then is the following: Is the total change of inclination of the tangents between any two points of the arch, or the relative displacements of those points, materially influenced by the web of the braced arch?

Charles Bender, C. E., has given, No. 26 of Van Nostrand's Science Series, an analytical treatment of the case of the deflections of a straight girder (simple or continuous). From his eq. 3, p. 82, we find, on substituting the values of $\gamma_1, \gamma_2, \ldots$, that the influence of the web in causing a change of inclination of end tangents is very small. This is approximately true for arches—flat arches especially. From eq. 4, p. 84, however, we see that angle x involving the deflections of a cantilever beam *is* influenced by the web. We should expect the same in an arch. It is easily proven by a careful construction thus: draw the arch with diagonal bracing, then starting at any apex construct the arch as changed from the tensions or compressions of its members, finding each successive apex in turn. If the shortening or lengthening of the pieces is exaggerated, we have an exaggerated view of the deformed rib. Now, if the same con-

struction be used when the diagonals are supposed unaltered in length, it will be found that the deformed rib differs materially in position from the first, and the more so the greater the depth of arch.

Stoney ("Strains in Girders," Plate 1) has given a similar construction for a straight truss. The difference in position of the two deformed girders, given by him in Plate 1, may be taken as a partial illustration of the bent arches given above. As a particular case, the pressure curve was assumed to pass through the apices of the lower flange, so that the change in the top flange was zero. The diagonals are also under no strain or very little. The exaggerated figure of the arch was then drawn; also the figure when the diagonals are supposed alternately lengthened and shortened, about one-third the corresponding change in lower flange.

The depth of arch was taken at about one-seventh of the radius; the half central angle being $72\frac{1}{2}°$, the semi-arch being divided into five panels. The departure from the free end was about $\frac{1}{6}$ greater for the case where the change in the diagonals was considered—certainly a very large increase to be neglected.

It is to be observed, however, that the pressure curve is variously inclined to the neutral line, and that the influence of the web when N acts up, is the opposite of that when N acts down; so that the error made in neglecting the web's influence may ultimately be small in consequence of this partial balancing of errors.

Again, if the sections of the web members are very large compared with usual strains, their alterations in length and influence in causing deflection becomes smaller in inverse proportion to the size of the pieces.

In the case of bridges with flat arches and small depth of arch ring, the influence of the open web diminishes greatly; but an actual construction for each case can alone determine the amount of error involved, which may thus be easily included in the factor of safety. For deep arches with small radius, however, such as some roof trusses, it is best to use a solid web; otherwise the theory hitherto given may not be applicable, and the strains become indeterminate. At any rate, it is well to test the influence of the web, for any. design, by an exaggerated construction of the deformed arch in order to include the error made under a proper factor of safety.

This objection does not apply to the braced arch with three hinges, since the theory of elasticity is not involved in establishing its true pressure curve.

87. The theory of elasticity is applicable, though in different degrees, in obtaining the true pressure curve, to *all* structures resting on two supports when a real or virtual change of space is considered. Thus, consider any form of girder or roof truss that fits perfectly when laid flat on its side, not then being subjected to any strain. When erected in place and the supports removed, the tendency is to push the abutments further apart, due generally to the compression of the upper members and the tension of the lower ones. If one end of the truss is on rollers, it slides, and the reactions are more nearly vertical, the smoother and more perfect the rollers or other device used. But if we suppose the abutments fixed in position, the reactions have to be determined by the condition that "the span is invariable." Even in a straight girder, where the lengthening of the lower chord is in part taken up by the curvature, the reactions are inclined slightly inwards; thus decreasing the strain in the lowest chord, whilst not adding to the strain in the upper chord. This lengthening of the span is thus on the side of safety, whether produced by the elastic elongations of the parts or by a rise of temperature over that at which the parts are fitted.

A fall of temperature, on the other hand, may cause the reactions to incline outwards.

Now, this tendency to a change of span is much greater in some structures than others, particularly in those of the abutting class. Thus, take the very graceful "crescent roof truss" with diagonal bracing: if one end rests on rollers it will practically act as a girder having vertical reactions; otherwise it becomes a braced arch, and its pressure curve is to be determined by previous conditions. The flanges in this case not being parallel, the formulæ of arts. 9–20 no longer apply.

Bow, in his "Economics of Construc-

tion," has called special attention to the above facts. On page 86 he gives an interesting method of finding the *true reactions* of a roof truss whose parts have been changed in length, due to the elasticity of the material. Thus, to take one case: Draw the diagram of forces for the reactions vertical, and compute the change of span, Δs, due to elasticity on the assumed sections. Next, find the change of span, $\Delta s'$, due to an assumed horizontal thrust, H. Since this change varies directly with H, we may now alter H in the ratio of Δs to $\Delta s'$, so that the total change of span is zero. The true reaction is compounded of the last value of H and the vertical reaction. Similarly, if the change of span is not zero, the final value of H must be such that $\Delta s - \Delta s' =$ assumed change of span.

It is suggested that the change of span due to elasticity can be readily found, as explained in the preceding article, from an exaggerated drawing of the deformed structure.

When the truss rests upon high, narrow abutments, their yielding is an unknown factor, so that it is well in such cases not to use a structure having much tendency to a change of span.

SUSPENSION SYSTEMS.

88. The theory hitherto exposed is equally applicable when the arches are suspended from the supports, the strains being changed in kind only. The suspended arch with three hinges has been lately used, and is peculiarly suited to overcome that want of rigidity observable in ordinary suspension bridges, besides being readily analyzed.

ARCHED ROOF TRUSSES.

89. The arched roof truss is more especially adapted to large spans. In this investigation, the flanges will be supposed concentric or parallel, so that the conditions of art. 9 *et seq.* apply; more especially if the web is solid or latticed, and approximately for a more open web.

When the weight of the structure, with any vertical loading, as snow, alone is considered, the analysis falls under that already given. If in addition, however, the wind is supposed to blow on one side of the structure, the case becomes one of oblique forces and is more difficult of solution.

90. If we suppose the wind to blow horizontally with a force P against a square foot of vertical plane, the experiments of Hutton go to show that the normal pressure on a square foot of roof surface inclined at an angle i to the horizontal equals

$$N = P \sin i^{1.84 \cos. i - 1}$$

More extended experiments are needed to verify this formula; but assuming it to be true, we have—taking P=40 lbs. per square foot, as the greatest intensity of the wind in a horizontal direction likely to occur, the following values of N in pounds for different inclinations of the roof surface, taken from Greene's Roof Trusses:

i	N	i	N	i	N
5°	5.2	25°	22.5	45°	36.1
10	9.6	30	26.5	50	38.1
15	14.	35	30.1	55	39.6
20	18.3	40	33.4	60	40.

For steeper slopes N is 40 lbs.

91. Let ACB, Fig. 15, represent the neutral line of a pointed roof truss; the arc AC being described with a center d, the arc BC with a center d'. In this figure each half arc is divided into 16 equal parts and ordinates drawn through the center of each division.

Assume the slope of roof i, on each division, to be that of a tangent to the center of each division, then the force of wind on the 15th division, inclined 80° to the horizon, is 40 lbs., and it acts in the direction of the radius $\overline{15, d}$. Hence lay off $\overline{dm}$ on the radius produced equal to 40 lbs, multiplied by area of division; if $\overline{dn}$ represents the weight of one division of roof, $\overline{nm}$ is the resultant oblique force acting at the center of division 15, through which a line is drawn parallel to $\overline{nm}$. The weight of each division, including purlins, sheeting, snow, &c., does not generally act through the center of the division of the neutral line; hence it is most accurate to combine N_1 with the vertical load acting through its center of gravity; so that the oblique line just drawn will be moved slightly to

the left. The subsequent constructions are the same in either case.

92. Proceeding in this manner for each division, values of N being interpolated from the table, when necessary, we next lay off on a force diagram on the left, beginning at e, the forces 1, 2, . . . , acting at the center of divisions 1, 2, . . . , in order and parallel to their directions. Similarly, on the vertical through e', lay off the equal weights on the divisions from C to B in order.

(The forces are here laid off to a smaller scale than that to which $\overline{nm}$ was drawn, for convenience).

The lines $\overline{ef}$ and $e'f'$ represent, therefore, the intensity and directions of the resultants of the forces just found from C to A and from C to B respectively.

The position of these resultants P_1 and P_2 are found as follows: The line eCe' was drawn in the first instance passing through the crown, at about the inclination it was thought the pressure curve would have there. Hence, assuming O and O′, equally distant from e and e', as poles, draw the pressure curves for the left and right halves of the arch respectively, starting at c with the assumed inclination of the thrust there. The curve on the right is drawn as usual and needs no explanation. It lies very near the neutral line at first and then passes above it. On prolonging the last side to intersection E′ with Ce', we find the position of the resultant P_2 of the forces on the right half of the arch.

For the left side, we extend Ce to intersection with force acting at 1, then draw a line ∥ ray 12 to force at 2, then a line ∥ ray 23 to force at 3 and so on. On prolonging the last side to E, we have $EP_1 \parallel ef$ the position of the resultant $P_1 = \overline{ef}$ of the oblique forces acting

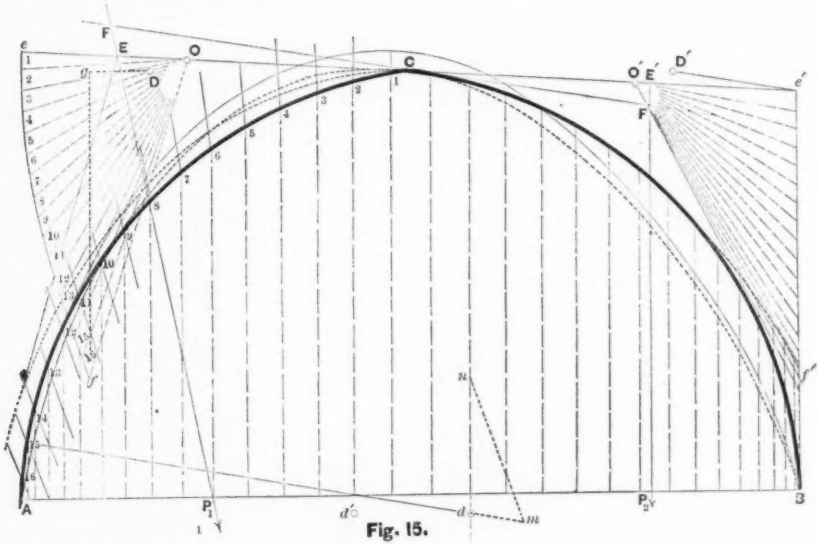


Fig. 15.

on the left half of the arch. If this were the true pressure curve, this "last side" would represent the *reaction* at A, which produced to intersection E with the thrust at C gives one point of the resultant P_1 as stated.

93. It is evident that so long as the loading remains the same that the positions and magnitudes of P_1 and P_2 remain the same for any pressure curve. It is now very easy to find the reaction at A in order that a pressure curve may pass through the three points A, C and B. Thus call the vertical and horizontal components of the reaction at A, V and Q respectively. On equating the moment of V with that of P_1 and P_2 about B, we find V

$$\therefore V.\overline{AB} = P_1 p_1 + P_2 p_2,$$

calling p_1 and p_2 the length of perpendiculars from B upon P_1 and P_2 respectively.

This supposes B to be at the same level as A, otherwise Q will have a moment about B. If P_1 is decomposed into vertical and horizontal components at the level of B, the term $P_1 p_1$ may be replaced by the product of its vertical component by its horizontal distance to B. To find Q take moments about the crown of the forces to the left of it. Thus calling the length of the perpendiculars from C to P_1 and AB, c_1 and h respectively, we have,

$$Qh = V\frac{AB}{2} - P_1 c_1$$

from whence Q is found.

Laying off, now $\overline{fg}=V$ and $\overline{gD}=Q$, we have D as the new pole on the left. On drawing $e'D'$ parallel and equal to eD, we have D′ for the position of the new pole for the forces acting on the right half of the arch.

The direction of the pressure at C is the line FcF' parallel to eD or $e'D'$.

Starting at C we draw the pressure curve as before. It is shown by the dotted line passing through A, C and B.

It is well to test the computed values of V and Q, before drawing the pressure curve, by drawing through A and B lines parallel to fD and $f'D'$. If these lines intersect FF′ at the same points F and F′ with P_1 and P_2, the poles D and D′ have been correctly found.

94. *The above pressure curve is the true one for the roof truss hinged at* A, C *and* B.

When the arch is not hinged at the abutments, the pressure curve will not generally pass through points at the same level at the springings. Thus suppose B of the new pressure curve to lie above A, and the point C to one side of the crown, and let it be required to pass a pressure curve through the new positions of A, B, C. First find the resultants (which call P_1 and P_2 as before) of the loads left and right of C (new positions of A, B, C, are intended in what follows). This is easily done by producing the proper sides of the equilibrium polygon to intersection, &c.

Denote the lever arms V, P_1 and P_2 about B, l, p_1, and p_2 respectively, the lever arms of V and P_1 about C, c and c_1 respectively. The vertical distances of C and B above A are h and b respectively. Taking moments now about B we have,

$$Vl = P_1 p_1 + P_2 p_2 + Qb$$

Again, taking moments of the forces to the left of C about C, we find,

$$Vc = P_1 c_1 + Qh$$

Eliminating Q from these equations, we get,

$$V = \frac{P_1(p_1 h - c_1 b) + P_2 p_2 h}{hl - cb}$$

Having found V from this equation, we can deduce Q from the preceding equation, and then proceed as before to draw the pressure curve through the assumed points A, B and C.

These formulae may also be employed in the case of a tunnel or culvert arch acted upon by oblique forces on both sides. See other formulae in "Voussoir Arches," art. 63.

95. Recurring to the pressure curve ACB, having the poles D and D′, we have the *bending moment* about any point on the neutral axis between 4 and 5, say, equal to ray D 45, measured to the scale of force multiplied by the perpendicular distance from the point to the resultant acting along the side 45 of the *pressure line.*

This moment is also equal to the horizontal component, H′ of ray D45, multiplied by the vertical distance v from the point to the resultant. This is evident, if we decompose the resultant at a point of its line of action vertically over the point taken in the neutral axis, into vertical and horizontal components. The latter alone causes a moment about the point equal to $H'v$ as stated.

It is well to note that v is not measured necessarily to the pressure curve, unless the resultant happens to act along the side vertically over the point. At a point where a force acts, as the one marked 4 of the neutral axis, the bending moment may be found by taking the moment of the resultant acting either side of the force. In this case if the resultant (=ray D45) to left of force is taken the moment=$H'v$, v being the vertical distance above 4 to the side of the pressure line between forces at 4 and 5, produced.

As the oblique forces are inclined in-

wards, it is seen that if we measure the distances v from 1, 2, . . . to the pressure line as usual, that when the pressure line is above the neutral line, H′ must be taken as the horizontal component of the resultant acting just to the *right* of the force acting at the point taken; otherwise, when the pressure line is below the neutral line, H′ corresponds to the resultant acting just to the left of the force.

Thus at 4, use the ray D34, at 13, the ray (D13, 14), in evaluating H′, v being measured vertically from 4 and 13 to the pressure line. As usual, v is + when above the neutral line, — below it, in finding ΣMy, etc.

96. For *the arch hinged at the ends, but continuous at the crown*, we have the conditions (E, I and s being constant) that the span is invariable,

$$\therefore \Sigma My = O$$

The moments M to the left of C are of the type H′v H′ and v both being variable; to the right of C, M=Hv, v alone being variable, H representing the constant pole distance from D′ to force line $e'f'$.

The ordinates y are, of course, measured from AB to the neutral line of the rib; and in the summation, v will be taken as plus when laid off above the neutral line, minus below it, since the moments M have different signs on opposite sides of the neutral line.

97. It is seen by inspection that the pressure curve passing through A, B and C must be raised at C to satisfy the eq. $\Sigma My=O$. Hence, a trial pressure curve was drawn, passing through A and B and a point about $\frac{1}{20}$ of an inch above C. Now measure to some scale (40 was used) the values of H′, v and y for each point 1, 2, . . . , for both halves of the arch and compute $\Sigma My=\Sigma H'vy$. It is found to equal in this case −1000. The curve should be raised slightly at the crown.

Now if we suppose applied at A a horizontal force Q′, and at B an equal force Q′, opposed to the other, we do not disturb equilibrium. The moment M′ at any point of the rib whose ordinate is y, due to Q′ is Q′y.

Now, if for an assumed thrust Q, we find that $\Sigma M'y=\Sigma Q'y^2=+1000$, it follows that by combining Q′ with the reactions previously found at A and B and drawing a new pressure curve, that we shall then find the resulting $\Sigma My=O$, since the moments caused by the resultant reaction must equal the sum of the moments caused by its components. To find Q′, we have simply to compute $\Sigma(y^2)=1500$ in this case; then since

$$Q'\Sigma y^2=\Sigma My \text{ (first found)}$$

it follows that $1500Q'=1000 \therefore Q'=\frac{2}{3}$.

This acts outwards in this case, since the curve has to be raised; hence lay off $Q=\frac{2}{3}$ from D in the direction Dg to find the true pole for the force polygon at the left, from whence the new position of D′ is found as before, and the pressure curve drawn as usual.

If the first ΣMy had been positive, then Q would act as a thrust, and the pole distance would be increased.

It is perceived that the vertical components of the reactions remain unchanged by the alteration in Q, as in fact follows from the preceding formula for computing V, in which no term involving Q occurs.

The true pressure curve, for the arch hinged at the ends, is shown by the full line in the figure, passing through A, B and a point nearly $\frac{1}{10}$ inch above C.

It is seen that this case offers no difficulty.

98. A tentative solution has been given by Wm. Bell, C. E. in his paper on "Rigid Arches, &c.," (see Van Nostrand's Magazine for May 1873), where the reader will find another method given of finding the reactions for a pressure curve assumed to pass through three points; also the graphical computation of the wind forces, according to the law of "the square of the sines."

In drawing the pressure curve for the arch "fixed at the ends," Mr. Bell has throughout neglected the condition $\Sigma Mx=o$, which, thus far, invalidates his results. The reader is referred to his paper for the case of the "rigid arch braced" on the supposition of hinged joints; which, even for straight rafters, represents the true *pressures* and *moments* at any point of the rafter for continuous loading, which is not so clearly seen when, as is usual, the loads are supposed concentrated at the apices. The writer has also illustrated this point in Science Series No. 12, Fig. 17. If the

loads are supposed concentrated at the apices, the longitudinal strain at the *middle* of the division of the rafter only is the same as that found from the consideration of continuous loading.

The bending moment there is, of course, the same, whether found graphically or computed for an inclined beam supported at the ends and continuously loaded.

Even for the case of curved rafters, the longitudinal strains, acting in the straight line joining two apices, due to the supposition of the loads being concentrated at the apices, may be found by the simple diagram of forces. The total strain at the center of the division of the rafter is then due to this strain, acting with its leverage to the neutral line, and the moment caused by the continuous loading on the part of the rafter considered "supported at the ends."

The supposition of hinged joints is not exactly realized for curved or straight rafters, however.

99. *Arched Roof Truss, fixed at the ends, continuous at the crown.*

The strains due to vertical loads may be found as hitherto illustrated in Fig. 6. An *exact* solution of the strains caused by the wind, is obtained by considering the wind force acting at each point *separately*. The case then becomes identical with that of an unsymmetrical arch acted on by a single vertical load, the arch being supposed tilted up at one end until the wind force, at the point taken, is vertical. The ordinates y must now be drawn parallel to the direction of the wind force (some of them may pass outside the arch), to the line joining the ends of the arch, which is taken as the axis of x.

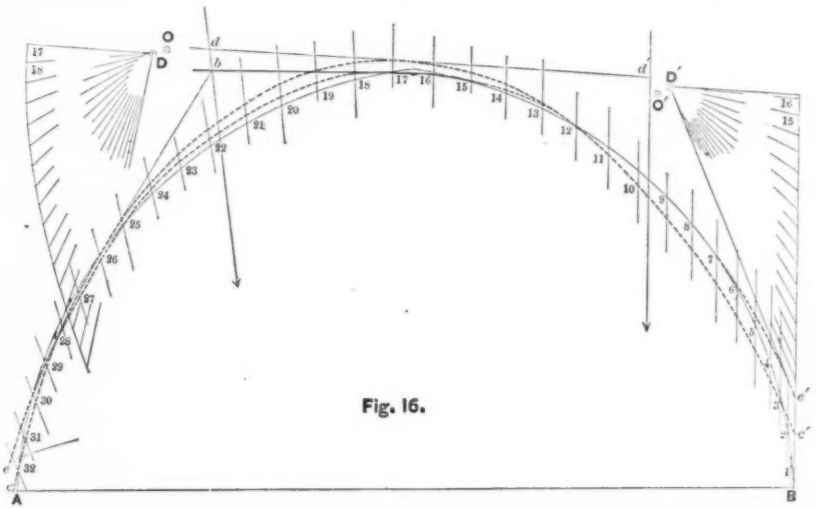


Fig. 16.

The conditions, $\Sigma M=o$, $\Sigma Mx=o$, $\Sigma My=o$, must be fulfilled, as is readily shown by the reasoning of arts. 12, *et seq.*; the first condition indicating that the tangents at the ends are fixed in position, the second and third, that the displacements of one end, relatively to the other, in directions perpendicular to x and y respectively, are zero.

The construction is now proceeded with as shown by Fig. 6, except that the closing line k_1k_2 of the arch is no longer parallel to the axis of x, but must be determined precisely as shown for curve p.

100. The general demonstration of art. 34 is especially useful here in indicating precisely the steps to take, as some of the ordinates, passing outside of the arch, appear to produce abnormal features.

Now it will doubtless be considered too tedious in practice to consider each wind force separately, hence a method that will give the strains due to wind force and vertical loading both, at one

operation, is especially desirable in a practical point of view. There are considerable difficulties in the way of an exact solution; but it is believed that the method given below, which leads by one or two approximations to a result very near the truth, is sufficient for the needs of the practitioner, especially in view of the fact that the theoretical conditions are not often realized in putting the structure together.

These conditions are—that the parts of the arch fit perfectly when under no strain, the fitting being tested by putting the parts together so that the arch lies horizontally, or on its side.

Again, the abutments should be connected with the arch by bolts; and should be so heavy that the bending moments at the feet of the arch will cause no appreciable rocking in them.

It is likewise a known fact that the wind pressure is greater as we ascend, especially if there are obstructions to its course near the base of the structure. Mr. Bell, in his attempted analysis of the roof of St. Pancras Station, has neglected the wind force near the abutments; possibly from the cause mentioned, though mainly, I presume, in consequence of the increase of section towards the abutments due to vertical walls, with a sort of spandrel filling between them and the arch proper, as well as the introduction of doors and windows in the part considered.

By neglecting the wind force on the lower part of the arch in the construction below, we are led to a more rapid approximation to the true pressure curve, so that it is in the line of simplicity.

101. Let us take A (Fig. 16, *on page* 453) as the origin of co-ordinates, x horizontal, y vertical. The true pressure curve for the arch fixed at the ends must then satisfy the conditions,

$$\Sigma M=o,\ \Sigma Mx=o,\ \Sigma My=o.$$

These conditions may also be written (art. 95),

$$\Sigma(H'\nu)=o,\ \Sigma(H'\nu x)=o,\ \Sigma(H'\nu y)=o.$$

Now let us draw a trial pressure curve as near the true one as possibly can be estimated. There is no guide from previous examples, as this case has never been correctly solved hitherto, but we may infer from the preceding equations, that the true pressure curve must cross the center line several times, and, moreover, must depart from it more on the left (H' being less) than on the right half of the arch.

The first trial curve was drawn through e, e' and a point slightly below the crown, as shown by the dotted line (on divisions 7 to 12 the curve coincides with the neutral line of the arch), the poles being O and O'. Call the constant pole distance for the right half of the arch H, the horizontal component of the resultant at any point of division on the left half of the arch H', and the ratio of H' to H, θ.

If we denote the vertical ordinates from AB to the arch and trial pressure curve, by y and y' respectively, we have at a point on the left of the center,

$$M=H'\nu=H\theta(y'-y)$$

It is sufficiently near to measure y' to the equilibrium polygon simply (and not to certain resultants produced as explained in art. 95) for the first construction.

102. The first two conditions above become,

$$\Sigma\theta(y'-y)=o,$$
$$\Sigma\theta(y'-y)x=o;$$

the summation being extended over the entire arch, θ being, of course, unity for the right half of the arch.

Next diminish the ordinates y and y' at each point of division on the left of the arch to θy and $\theta y'$, using the value of θ corresponding to the point, and lay off the altered curves as shown in Fig. 17, the right half of the arch, &c., being the same as before. The values of H' are readily found by measuring, to scale, the horizontal distances from the points of division of the force line to a vertical drawn through O. The altered lengths may be easily found by a slide rule or by the usual ratio lines.

103. Let us denote the ordinates from a straight line k_1k_2 (Fig. 17) to the altered curve of the neutral line by a, and draw k_1k_2 so that $\Sigma a=o$, $\Sigma ax=o$. The construction is similar to that used in finding the closing line mm' of polygon p in Fig. 6.

If we denote the ordinates from k_1k_2 to the pressure curve by c, the above equations become,

$$\Sigma(\theta y' - \theta y) = \Sigma(c-a) = o$$
$$\Sigma(\theta y' - \theta y)x = \Sigma(c-a)x = o$$

Now since $\Sigma a = o$, $\Sigma ax = o$, it follows that when the pressure curve is adjusted to its right position that $\Sigma c = o$, $\Sigma cx = o$ also. Therefore, calling the ordinates from a straight line $m_1 m_2$ to pressure curve, c', $m_1 m_2$ being so drawn as to satisfy the conditions $\Sigma c' = o$, $\Sigma c' x = o$, we now conceive $m_1 m_2$ placed over k_1 k_2 and the ordinates c' laid off above and below $k_1 k_2$ to the true position of a pressure curve c, whose ordinates now satisfy the conditions $\Sigma(c-a) = o$, $\Sigma(c-a)x = o$.

104. We have now introduced one approximation, viz: that the ordinates of an equilibrium polygon, due to oblique forces in part, are unchanged in length by the particular position of the pole, so that it lies in the same vertical. This would be correct if the vertical *reactions* at the abutments of a supposed girder, acted on by the same external forces, were always found in the same vertical, since the bending moment at any point is then *constant* and equal to $H'y''$, y'' being the ordinate from the closing line to the curve. It is evident that since H' is constant, that y'' must also remain the same for any pressure curve drawn with the same pole distances.

In this case, however, the point e will move along the inclined force at 32 (Fig. 16), as the pole o is moved up or down vertically, so that the vertical component of the reaction will not remain in the same vertical, therefore the moment is not constant. Still the result will not vary much from the truth, and would be exact, so far, if the wind forces near A were discarded.

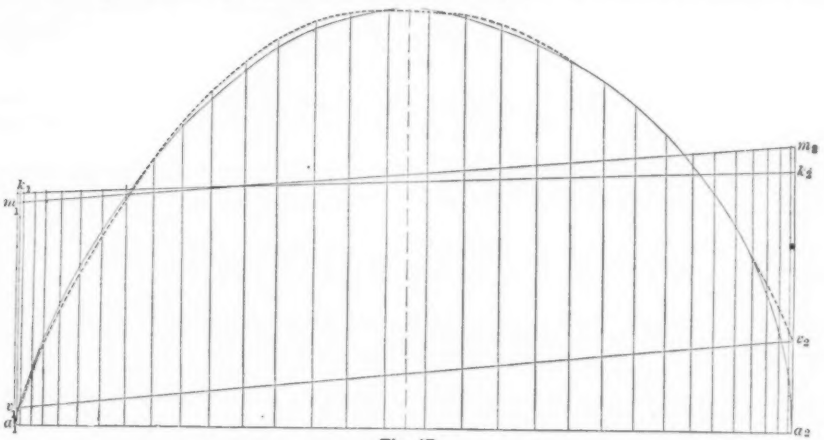


Fig. 17.

105. We come now to the third condition, $\Sigma My = o$

$$\therefore \Sigma\theta(y' - y)y = o = \Sigma(\theta y' - \theta y)y = o$$
$$\therefore \Sigma(c-a)y = o,$$

or $$\Sigma cy = \Sigma ay,$$

c representing the ordinates from $m_1 m_2$ to dotted curve Fig. 17, a the ordinates from $k_1 k_2$ to curve a Fig. 17.

Find Σcy and Σay as in Fig. 7. This construction is readily made on Fig. 17, by drawing horizontals through the tops of the ordinates y for both halves of the arch, laying off the ordinates c and a as forces on a horizontal, and proceeding exactly as in Fig. 7. If the above equality does not hold, change the lengths c in the ratio of Σcy to Σay to cause the equality, and lay off the new values of c above or below $k_1 k_2$ (the ultimate position of $m_1 m_2$) at several points. The curve so drawn crosses curve a at points corresponding to those marked 12 and 26 in Fig. 16, c' being a third point in the proposed pressure curve, which may now be drawn through c' 12, and 26 of Fig. 16, as shown by the dotted line passing a little above A and the crown of the arch, and through the three points mentioned. This curve is described with

the poles D and D′, and is evidently very near the true one, since the pole distances have been but slightly changed. If, starting with this curve, with the new values of H′, &c., we repeat the preceding construction, it is plain that the resulting pressure curve will be still nearer the exact one, and will probably differ from it by an inappreciable amount, practically considered. On this small scale it is of course useless to make the second construction. The result is readily tested from the given conditions.

The first construction would be the correct one if the ordinates c could all be changed in the same ratio due to a change of pole distance (as for vertical forces), but this not being exactly true for the left half of the arch, a change in pole distance altering the ratios θ likewise, so that the positions of k_1k_2 and m_1m_2 would not remain the same, it is evident that the true pressure curve is not exactly found. The smaller the wind forces, compared with the vertical loads, the nearer the first approximation to the truth. It would seem that the best result is obtained, by passing the final pressure curve through the points of intersection (12 and 26) of curve c with a Fig. 17, m_1 m_2 coinciding with k_1k_2, as was done in this case.

106. The position D of the pole may be found very readily in a tentative way thus: let it be required to pass a curve of pressures through c', 26 and the point over the crown shown by the upper dotted line. By producing the sides $\overline{16\text{-}17}$ and $\overline{26\text{-}27}$ of any pressure curve to intersection b Fig. 16, and drawing a line parallel to the resultant of the forces from the crown to 26, as obtained from the force line on the left, we have this resultant, which pases through b, determined in position and direction.

Now draw some line dd', supposed to have about the position of the side of the equilibrium polygon at the crown, to intersection d and d' with the resultant just drawn and the resultant of the loads to the right of the crown. Then through the lowest point of the force line on the right draw $\overline{e'\text{D}} \parallel \overline{c'd'}$, on the left draw ray $\overline{26\text{-}27} \parallel \overline{26.d}$.

Now from the tops of each force line draw lines parallel to $\overline{dd.'}$ If these lines intersect the rays previously drawn, in points D and D′, the distances cut off being the same, then the line $\overline{dd'}$ was correctly located. Two or three trials generally suffice to establish the poles correctly. To pass the curve through 12 26 and c', we find the resultants either side of 12 and proceed as before. It will be found best, in this case, to lay off the forces on *one* force polygon, in place of two as hitherto. It may be observed, however, that in a large drawing, especially, it is a great convenience to have the force polygons, as in the figure, directly over the part of the arch to which they apply.

107. The method of drawing the pressure curve detailed above possesses the merit of simplicity, but it is not so accurate as the one given in "Voussoir Arches," art. 67, since any error made—and some error is incident to any construction—is carried on, which is not the case in the following method.

At each point 1, 2, . . . , 32, resolve the load into vertical and horizontal components; find the moments about the crown of each component. Now if we consider any number of consecutive loads from the crown towards either abutment, we have only to divide the sum of the moments of the vertical components by the sum of those components to find the distance to the resultant of the vertical components from the crown, which lay off. Similarly find and lay off the vertical distance below the crown of the horizontal components. Now on drawing a vertical through the first point found and a horizontal through the second, their intersection gives a point in the resultant of all the loads acting on the part considered, which can now be drawn in position parallel to the proper direction taken from the force diagram.

Thus we may find the resultant of all the forces acting from the crown to 26 inclusive, as shown by the arrow passing through b.

If the direction of the thrust dd' acting at the crown is given, as well as its position from its intersection d with the resultant just found, we have simply to draw a line ‖ray D 26, 27, to find the center of pressure at 26, or more accurately speaking, about midway between 26 and 27. The same method applies to each

division in turn. On the right half of the arch the construction is simplified, since here we only have vertical forces.

If this method is not pursued, then the positions of P_1, P_2, etc. (Fig 15), should be found by at least two independent constructions, whence the true positions of the centers of pressure at the abutments may be at once found, from an assumed thrust at the crown, thus affording a check upon subsequent constructions.

108. It is evident from the considerations of art. 35 *et seq.* that the pressure curve of an *underground voussoir arch* is identical with that for the solid arch, when no joints tend to open and the mortar is thin and hard. Since in this case there are horizontal forces on both sides of the arch, the pressure curve will depart from the center line about the same distance on either side of it at the points of maximum departure.

109. When EI is not constant for the entire arch, the term θ may be taken to represent $\frac{H'}{EIH}$ (see art. 41), when the construction will proceed as before, the proper values of θ being computed for each point of division of the arch.

The strains due to change of span, temperature, etc., are readily and correctly computed as hitherto explained. It is evident, therefore, that if the arched roof truss is built in accordance with the hypotheses, that the strains, due to whatever causes, may all be found.

110. It is seen from the foregoing how well adapted the graphical method is to the investigation of the strength and stability of every form of arch—many forms being almost intractable by ordinary analysis—presenting likewise the great advantage of keeping prominently before the worker the very hypotheses upon which the theory is built.

If the writer has added materially to the knowledge on the subject of the "Solid or Braced Elastic Arch," or has aided the student in its comprehension, he feels amply repaid for the time spent in developing the foregoing theory.

THE DYNAMICAL POWER OF STEAM.

From "Iron."

At a time like the present, when economy in every department of mechanics and manufacture is eagerly sought after, it is somewhat strange that no efforts should be made to utilize, on a large scale, the dynamical energy of steam. This is, perhaps, owing to the fact that the form of dynamical energy in steam is not much known or recognized beyond its limited application to injectors and ejectors. The work to be obtained from steam, whether used statically or dynamically, may be compared, theoretically, very considerably to the disadvantage of the former method. The statical use of steam is the system at present represented by a reciprocating piston in a cylinder, where the expansive pressure of the steam is alone used, the velocity of admission to the cylinder being rendered purposely as moderate as possible. The condition of dynamical energy would be that of an escaping jet. It may be thought, at a first glance, that an escaping jet of steam represents no useful condition for work; but this idea is evidently a fallacy, as this form of energy is very advantageously made use of in the injector or ejector pump. In this class of instrument there cannot possibly exist any form of statical energy, as they are invariably constructed with an overflow orifice open to the atmosphere. The condition of work here is evidently a certain weight of steam per minute rushing into the atmosphere, or even into partial vacuo, with a great velocity, and thus inducing or drawing to itself a much heavier weight of water, which, mingling with the steam, acquires a portion of its velocity in inverse ratios to their respective weights, and at the same time producing a partial vacuum which tends to accelerate the issuing jet of steam. The resulting volume of water and steam has thereby acquired a momentum which is sufficient to overcome the statical resistance of the check valve and the boiler pressure behind it.

This condition of matters is highly in-

structive from the point of view that it is thus demonstrated to be possible, that steam generated under statical pressure, may, by combination with a volume of water much heavier than itself, be rendered capable of overcoming its own statical resistance. It is also a self evident axiom, that the total momentum of the effective column of water and steam will remain the same as that of the original escaping jet of steam, less loss by friction. The conclusion is, therefore, perfectly logical that the total dynamical energy of the escaping jet of steam was perfectly capable of overcoming the original statical pressure of its generation. This is a curious point for further investigation. It is not difficult to make an arithmetical comparison between the dynamical energy of a given weight of steam and the known work usually obtained from it, by the ordinary method of working in the cylinder of a steam engine. First, the velocity of an escaping jet of steam into vacuo may be calculated in the same manner as that of an escaping jet of water, and from the formula $v=\sqrt{2gh}$, where "v" is the velocity in feet per second, "g" is "gravity," and "h" is the imaginary head in feet of the fluid in question, of incompressible density, that would suffice to produce the given pressure of investigation per square inch on the orifice. As steam at atmospheric pressure is about 1700 times lighter than water, "h" would represent a value of about 59,000 for each atmosphere of pressure. For four atmospheres of pressure $v=\sqrt{2\times32\times236,000}=15,104,000$, and the whole actual energy of the steam may be represented by its

$$vis\ viva=\frac{Wv^2}{2g}=236,000 \text{ foot-lbs. per lb.}$$

of steam. But it is well-known from practice that about 23 lbs. of steam are required per hour to indicate one horse-power per minute in the best class of engines$=33,000\times60=1,980,000$ foot-lbs. per hour=about 86,000 foot-lbs. per lb. of steam, or rather more than $\frac{1}{3}$ of the above expressed dynamical energy.

The effective value of the dynamic power of steam having thus been fully shown, it only remains for the practical engineer to devise, if possible, a feasible method for its utilization. We have previously observed that it is already used for the injection or ejection of liquids, where the principle is made use of, that steam of exceedingly light weight, but very high velocity, may be made to set in motion a much greater weight at a reduced velocity, so that the resulting momentum remains the same. So far the principle of dynamic force has been fully utilized, but we have stopped its application at that point, and have not been able as yet to apply such a motive power to produce constant rotation, which is the special development of force most generally required for industrial purposes. We have heard a rumor lately of a French invention which proposes to effect this development in the sense of producing a constant motion from a dynamic development of the elastic force of steam. For motive purposes this utilization can only be in the form of a moving column of water, which by its greater weight and reduced speed may be possibly converted into an active and constant power-producer. From our view of the subject, however, it will be difficult to construct any mechanical appliance to utilize such a moving stream, which shall not at the same time waste so much by leakage and friction as to spoil its economical effect. In seeking such a mechanical utilization of dynamical power, we should approach very closely again the forms of continuous rotary engine which have so long been unattainable in an economical form.

We believe, however, that in this class of motor it has never been attempted to utilize a moving fluid denser than steam, such as water, and which latter, we would observe, may be just as easily set in rapid motion as steam itself. Now, the utilization of a rapidly-moving stream of water is not such a difficult problem. It is already done every day with an advantage of 75 per cent. effective out of the moving power. If, therefore, our calculations be correct, as to the dynamic force of an issuing stream of steam, and if this jet of steam, at an enormous velocity, can be converted by an amalgamation into an immensely heavier and slower stream of water, we may assume, even on this basis, an effective utilization of the power of steam, some two and a quarter times greater than by the statical method at present in use.

LINKAGES.

A KINEMATIC DISCUSSION OF THE DIFFERENT FORMS OF ARTICULATED LINKS WITH ESPECIAL REFERENCE TO PEAUCELLIER'S CELL.

By J. D. C. DE ROOS.

Translated from "Revue Universelle des Mines" for VAN NOSTRAND'S MAGAZINE.

V.

In seeking for the inverse value of z or O″E by the reciprocater DD′(Fig. 42) we obtain for the equation of the curve described by F:

$$O''F=\frac{C^2}{z}=\frac{1}{\frac{p}{q}\frac{(w+q)}{C^2}\cos\varphi\pm\sqrt{\left\{\frac{(q-w)^2}{C^4}-\frac{(q-w)^2}{C^4}\frac{p^2}{q^2}\sin^2\varphi\right\}}}$$

But the ellipse and the hyperbola of which the axes were a and b have an equation of the same form, supposing that the origin is situated upon the prolongation of the principal axis at a distance $O''G=v-a$ (Fig. 42) beyond the vertex, or at a distance v from the center.

In effect, in this case, the general equation becomes:

$$z'=\frac{1}{\frac{v}{v^2-a^2}\cos\varphi\pm\sqrt{\left\{\frac{a^2}{(v^2-a^2)^2}-\frac{a^2[b^2\pm(v^2-a^2)]}{b^2(v^2-a^2)^2}\sin^2\varphi\right\}}}$$

The positive sign of the radical relates to the ellipse, and the negative sign to the hyperbola, while the parabola whose parameter is b, and whose vertex is situated at a distance $O''G=v$ from the origin O″ has for an equation,

$$z_1=\frac{1}{\frac{1}{2v}\cos\varphi\pm\sqrt{\left\{\frac{1}{4v^2}-\frac{l+4v}{4lv^2}\sin^2\varphi\right\}}}.$$

It follows that the point F describes

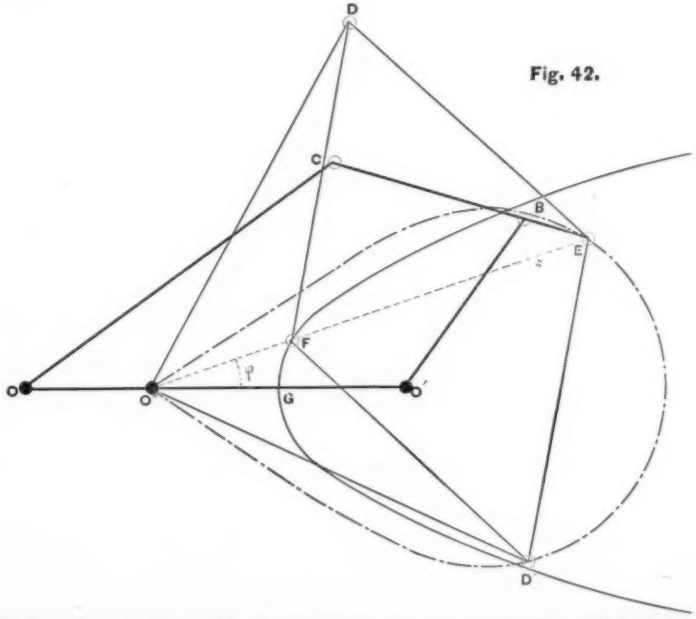


Fig. 42.

a conic section FB and that consequently the system of links devised by Roberts is a conicograph having three fixed points.

By equating the co-efficients, we find for the ellipse and the hyperbola

$$\frac{(w+q)}{C^2}\cdot\frac{p}{q}=\frac{v}{v^2-a^2},\qquad \frac{(q-w)}{C^2}=\frac{a}{v^2-a^2}$$

and $$\frac{(q-w)^2}{C^4}\cdot\frac{p^2}{q^2}=\frac{a^2[b^2\pm(v^2-a^2)]}{b^2(v^2-a^2)};$$

for the parabola $$\frac{1}{2v}=\frac{(w+q)p}{C^2q},\quad \frac{1}{2v}=\frac{q-w}{C^2}$$

and $$\frac{l+4v}{4lv^2}=\frac{(q-w)^2}{C^4}\cdot\frac{p^2}{q^2}.$$

The first case gives rise to three equations with four unknown quantities a,b,v, and C, of which one must consequently be chosen arbitrarily.

In supposing v known the axes are:

$$2a=\frac{2qv(q-w)}{p(w+q)}$$

and

$$2b=2qv\sqrt{\pm\frac{p^2(q+w)^2-q^2(q-w)^2}{p^2-q^2}},$$

and the power

$$C^2=\frac{v(q+w)[p^2(q+w)^2-q^2(q-w)^2]}{pq(q+w)^2};$$

If, on the contrary, the power is given, we have

$$2a=\frac{2(q-w)q^2C^2}{p^2(q+w^2)-q^2(q-w)^2},$$

$$2b=\frac{2q^2C^2}{\sqrt{\pm(p^2-q^2)[p^2(q+w)^2-q^2(q-w)^2]}},$$

and the distance $O''G+a$ from the origin at the center of the ellipse or hyperbola is

$$v=\frac{pq(w+q)C^2}{p^2(q+w)^2-q^2(q-w)^2};$$

the positive sign belongs to the ellipse, and the negative to the hyperbola.

In the second case we have

$$w=\frac{q(q-p)}{p+q},$$

$$v=\frac{l(p^2-q^2)}{4q^2}\text{ and }C^2=l(p-q)\frac{p}{q}$$

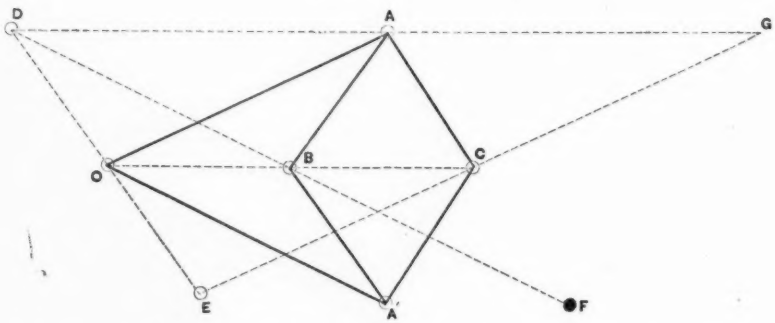
or the parameter

$$l=\frac{C^2q}{p(p-q)}.$$

In figure 42, $p=6$ and $q=3$ centimeter, while $v=2$ centimeters, which renders w negative and $=-1$ centimeter, and gives for the parameter a length of $2\frac{2}{3}$ centimeters; in which case $C^2=16$.

Then it results from the values of v

Fig. 43.



that in the conicograph of Roberts, the origin O'' can never be either at the vertex or center of the curve.

Nevertheless this conicograph is to be preferred to the four preceding ones, especially because of the small number of links required, and which may even be still further reduced to seven.

The same simplification which was applied to the pantograph of six bars, Fig. 7, to produce the pantograph of four (Fig. 8) can equally be applied to the *reciprocator*, although not in so ready a manner.

Through the point C′ of the positive cell AA′ (Fig. 43) draw a right line CE

parallel to OA, and from the point O a line OE parallel to AC; then prolonging EC and EO so that EG=2EC, and ED=2OE, we shall have the points D, A and G in the same right line, (just as in the pantograph Fig. 8 the points O, E′ and B fell), and also AD=AG.

From the point D draw the right line DF passing through B, and make DF=2DB; since DO is equal and parallel to BA′, DB must be equal to OA′; that is to say, it preserves an invariable length.

The same may be said of CE and OE, so that CA′, A′B and A′O can, without modifying the movement of B, be replaced by the links CE, ED and OD.

On the other hand the triangle DBA is similar to the triangle DFG, and consequently FG=2AB; also of invariable length, so that the three remaining bars of the losenge OA, AB and AC can be suppressed by uniting F to G.

This leads us to the combination shown in figure 44, having three fixed points O, B and C situated on a right line parallel to the diagonals DG and EF, while the four links are in pairs of equal length.

The relation between the arms OB and OC is deduced immediately from Figs. 43 and 44 which give:

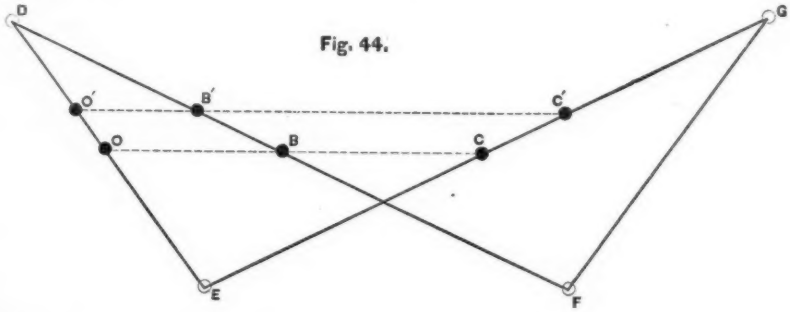


Fig. 44.

$$OB \cdot OC = OA'^2 - AC^2 = BF^2 - DO^2$$
$$= \tfrac{1}{4}(DF^2 - DE^2)$$

while for the arbitrary arms O′B′ and O′C′ we have because

$$DO' : \tfrac{1}{2}DE :: O'B' : OB$$

and

$$EO' : \tfrac{1}{2}DE :: O'C' : OC$$

therefore

$$O'B' \cdot OC' = 4OB \cdot OC \cdot \frac{DO' \cdot EO'}{DE^2}$$
$$= \frac{DO' \cdot EO'}{DE^2}(DF^2 - DE^2)$$

It was Hart who first noticed this property, and gave a direct demonstration of it.

We may then, in the preceding constructions, replace the reciprocator of Peaucellier by that of Hart, which we will describe, and which affords the transformation from alternating circular to alternating rectilinear motion, by means of five links instead of seven.

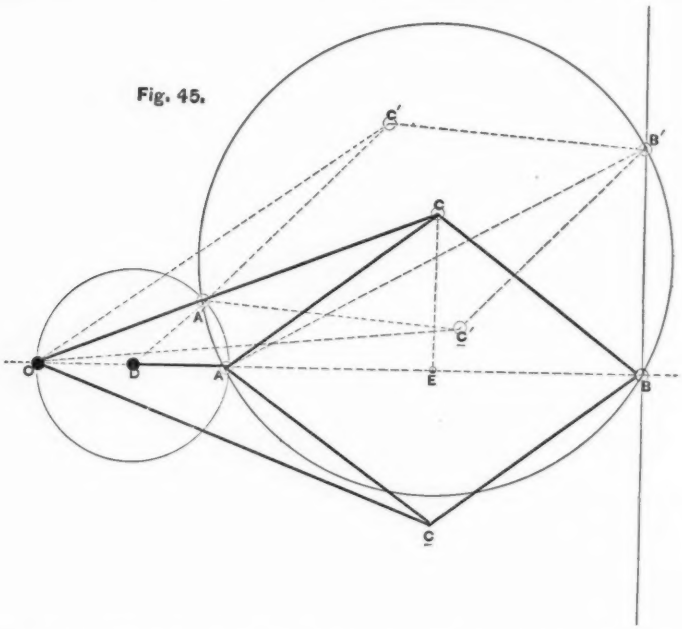
There is, however, no great practical utility in this property, for if the number of links is diminished by two, the number of independent attached points is increased by two whenever the apparatus is employed in combination with others.

Moreover, any point situated on ED or fixed to ED as is O′ in Fig, 44, describes relatively to FG (or EG) an inverse conic section, so that this element, like that of Roberts, can serve as a conicograph.

The particular movement of this system (or if preferred of the two elements of Hart) can easily be derived from the elements of Roberts. We find that in the element of Hart, the motion depending upon the crossing or non-crossing of the sides of the isosceles trapezeum, is like that of the Roberts element under the conditions of $q > p$ or $q < p$ respectively.

In this case, however, the polar lines of the curve O′ are no longer ovals of Descartes, but equal ellipses, whose major axes coincide with ED and FG

Fig. 45.



and are equal to DF and EG, while the foci are at D and E for one and at F and G for the other. Or, again, the two polar curves may be two equal hyperbolas of which the larger axes DF and EG coincide, and of which the foci are at D and F, or at E and G.

To these examples might be added a great many others, but we believe these to be sufficient to prove the great practical utility of the different systems of articulated linkages.

Until recently the number of curves that could be traced mechanically was quite limited; among the conic sections the ellipse alone was described by the elliptical compass, and by the apparatus of Leonardo da Vinci; leaving out the trace obtained by the tense thread.

The two methods referred to are based on the property—that all points of a straight line of determinate length describe ellipses when the extremities of the line are made to slide along the lines bounding a right angle. These devices have been extensively used in practical draughting, and have been regarded as indispensable.

Da Vinci's apparatus is described in many works; among others in Hartmann's *Metalldreherei;* in Thon's *Drehkunst*, and in the more recent work on Kinematics by Reuleaux.

The oval obtained by the universal lathe of Kochand Muller, in which the friction, and consequently the wear, is much less than in Da Vinci's apparatus, is not quite symmetrical, and is therefore not an ellipse.

The linkages for describing the other conic sections will find, without doubt, applications as extensive as those above stated. The parabola, for example, is the proper form for a reflector, and for the outline of solids of equal resistance throughout. Furthermore, the parabola may, from an æsthetic point of view, profitably replace the arc of the circle in our machines and other constructions, as is urged by Reuleaux.

The systems of linkages are, in our opinion, of great value, both from the theoretical and practical point of view.

The following memoirs have already appeared upon this subject: *Des systemes articulés simple et multiples, et de leur applications*, by Saint Loup,

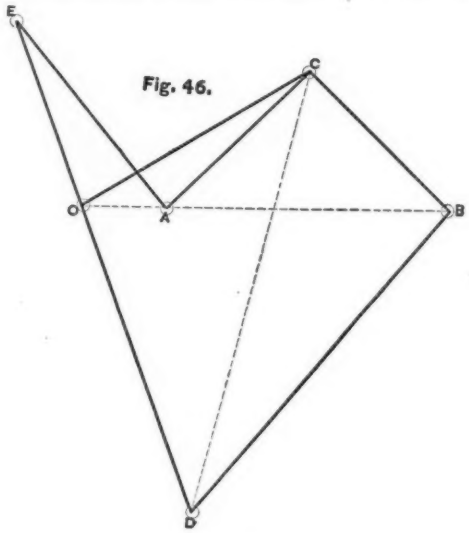


Fig. 46.

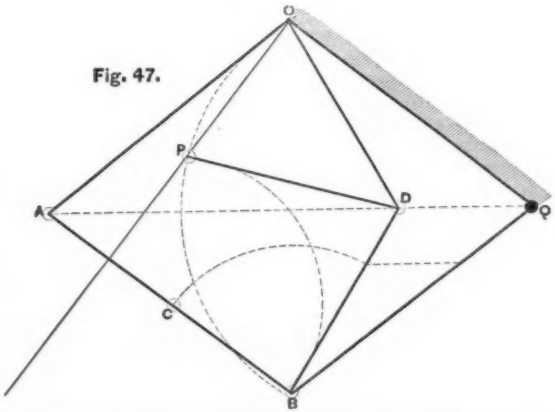


Fig. 47.

and, *Sur les systemes de tiges articulés*, by Liguine.

We may remark here that Liguine, who departs from the plan of the system of Fig. 9, urges that it is not indispensable, that the bars should be in pairs of equal length, in order that the points O, B and C remain in a right line. He says that it will suffice if the diagonals AA′ and II′ are perpendiculars upon OC(Fig. 50); which is an immediate consequence of the fact that the formulas contain, not the absolute values of the sides, but the difference of their squares, as has been already mentioned.

But this restriction of Liguine is too particular, since in any quadrilateral OBCD (Fig. 46), whose diagonals may intersect at any angle, a point A of a diagonal can always be taken so related to two of the sides that O, A and B shall be in a line, and so that the product of OA and BA shall be constant. We propose to demonstrate this case in a future note, in which we will give special attention to this new element, and to some other systems, such as that of Kemp, represented in Fig. 47, and in which P describes a perpendicular PO to the line OB at O; which results whenever

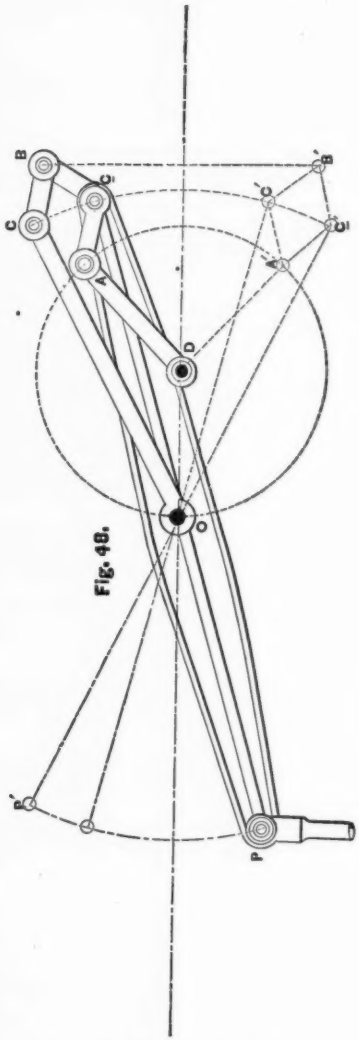


Fig. 48.

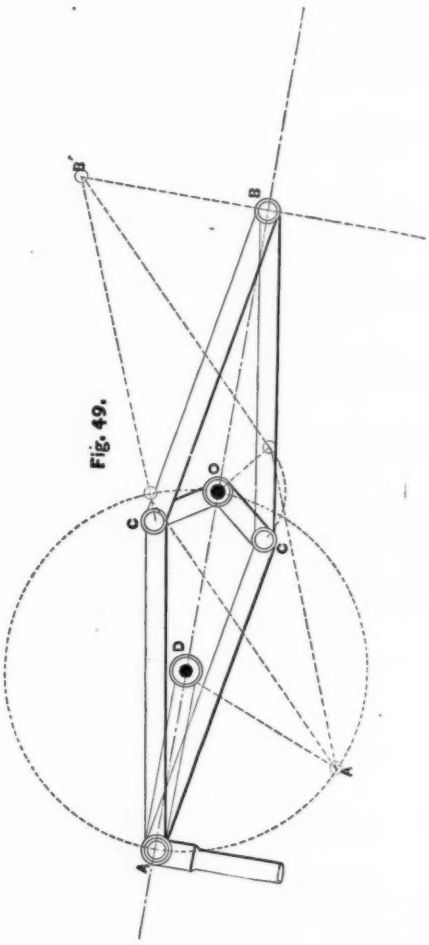


Fig. 49.

$$OA=AB=BQ=QO$$

$$OD=DP=DB$$

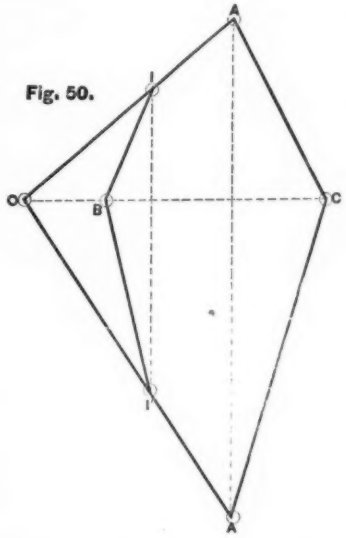
and $$PC=BC=\frac{OD^2}{OQ}$$

[NOTE BY THE FRENCH TRANSLATOR.]

The interest attaching to the beautiful invention of Peaucellier is incontestable, either from the kinematic or mathematical point of view. But we may nevertheless be permitted to remark that the complexity of some of the solutions neutralizes their beauty, and renders doubtful their utility.

In mechanism simplicity is of the first importance, and ought always to be one of the principal factors of a new invention. For instance, in one example we have seen that the extraction of the square root by means of linkages re-

Fig. 50.



quires the combination of three *eléments variés*, and four ordinary Peaucellier elements; altogether forty-two links or bars.

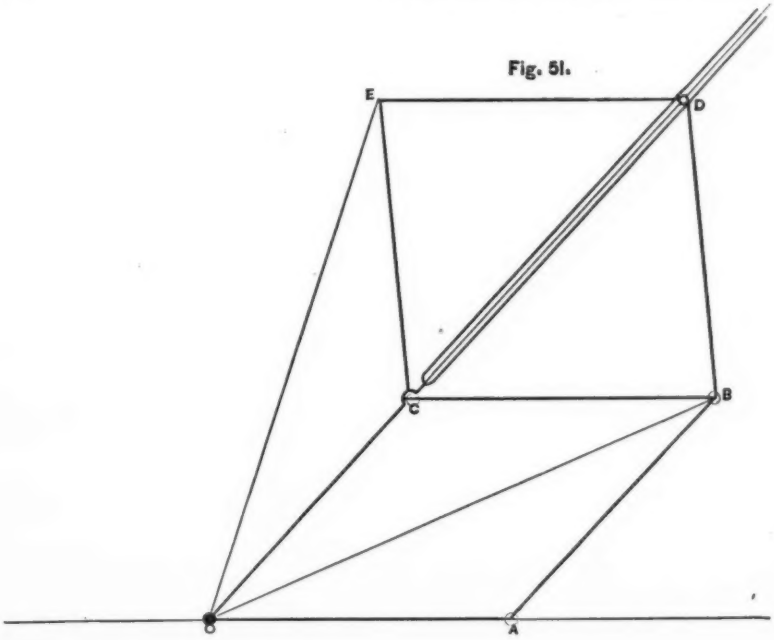
It is very probable, as the author remarks, that this combination may be much simplified. (Since a parabola can be traced by a conicograph relatively simple, we may employ the latter for extracting the square root by making the parameter equal to unity, for then $y^2=x$ and $y=\sqrt{x}$).

But we do not wish to criticise here this particular case, but we may call attention to the dangers of the tendency of mathematicians to enjoy such difficulties; a tendency for which the complex combinations of the Peaucellier cell affords great opportunities for development.

One of the most direct applications of this cell is its substitution for the parallelogram of Watt. Figs. 48 and 49 exhibit the mode of application to the piston-rod of a steam-engine. In Fig. 48 the fixed point O coincides with the center of motion of the beam, and D is a fixed point independent of the beam. B is fastened to the piston-rod. The walking-beam is the seventh bar OC. This is a positive element.

A better application is shown in Fig. 49, since the losenge itself forms the

Fig. 51.



walking-beam, having O for a center of motion, and D as the independent fixed point.

The combinations described by the author for the mechanical solution of equations of the higher degrees are, without doubt, very ingenious, and we feel convinced that if some one would make them for sale, many persons would find them of great use.

Referring to the generalization that the author makes of the problem of rectilinear movement, by means of linkages, and which leads to the new element of Fig. 46, we propose for it the name of the "Element of De Roos" in honor of the inventor.

THE VENTILATION OF SEWERS.

By JOHN G. WINTON.

From "Journal of the Society of Arts."

The furnace system for the ventilation of coal mines has now become obsolete. The plan was a sluggish one, depending on the rarefaction of the air in the furnace or upcast shaft, which was so arranged with those intricate passages in the depths of the earth, that, as the rarefied air ascended, another or downcast shaft supplied fresh air to the workings. The suction fan now takes the place of the furnace system. In some instances air is discharged and supplied by these fans at the rate of 100,000 cubic feet and upwards per minute.

Suction fans have been condemned by some authorities as inapplicable for the ventilation of our main-drainage systems. I think differently, that the main-drainage systems of Great Britain have been constructed in ignorance as regards ventilation. Our forefathers, no doubt, did not experience the want of proper ventilation and dilution of sewer gas which now exists in densely-populated cities. Modern works have been constructed, ventilating shafts have been adopted, forming upcast and downcast, according to the locality, nature has been coaxed to a rate these foul receptacles, the sluggish gasses have been purified by the charcoal process, and the noxious vapors have been trapped from our dwellings; yet, with all these precautions, very little has been done to create an in-draught into our main sewers, dispersing the vapors from the subways far overhead.

In the first instance, I consider that the sewerage works of towns should be so planned that the low-lying districts are kept separate, as far as practicable, from those of the higher districts; that each district should have a separate sewer, carrying away the sewage and rainfall from that district alone, they should debouch into a main common to all, and should be arranged with large siphon bends, so that the sewer gas of one district may not flow into that of another district; thus I may be able to deal most effectually with the sewer gas. With such an arrangement, I propose drawing the gases out of each district, discharging them into the main common to all, and which eventually would be discharged at the out-fall, or made to pass up a chimney, and be dispersed far overhead.

Secondly, that all the existing water-carriage plans for the removal of refuge are defective, in the absence of an abundant supply of flushing water directly applied to the main drainage system. I have advocated pumping up sea-water for this purpose for sea-coast towns, and, indeed, for all towns where it could be cheaply and conveniently applied. I consider that, by the application of sea-water, the refuse would be pickled, and would find its way seaward before decomposition took place, and noxious gases were evolved. In all other towns the fresh water supply should be sufficient to meet ordinary requirements, as likewise for periodical or continuous flushing. In the application I have simply considered it necessary to flush the house pipes; and when these tributaries

are promptly flushed, the mains will likewise be so.

If it is once ceded that continuous flushing is needed for the present system, I consider that a separate system, independent of rainfall, is preferable for the removal of refuse, and the effluent water from our dwellings. We should be able to dispense with gullies, and, as there would not be so many air-holes, the gases would be more effectually dealt with. However, there can be no objection to carry away the rainfall, discharging it into the large main already mentioned, which finally terminates at the outfall.

The small jets of water for flushing the pipes leading from the houses need not be more than $\frac{1}{2}$ inch in diameter; the water should be made to spread, striking against the side of the pipe, thus tending to prevent in-draught, and placed in such a position as not to be affected with refuse. For sake of illustration, 4,000 of these jets would represent an area of 784 square inches, or a diameter of nearly $31\frac{1}{2}$ inches; so, with the water delivered under a moderate pressure, as from ordinary cisterns, these small tributaries would eventually create a great rush of flushing water in the mains, the delivery being carefully calculated to suit the requirements.

In some recent examples for the ventilation of sewers, open gratings have been fitted in the middle of the roadways, in connection with shafts leading from the top of the main sewers. Many towns have such an arrangement. These, no doubt, allow the gases to flow out of the mains, more especially when the sewer gases are compressed by a sudden flow of water. In warm weather, with a minimum flow of water through the sewers, the gases rise very sluggishly, and we know the exhalations are very offensive. The "sun" is the furnace, as it were (as in the plan adopted by the early miner); and the sewers being of lower temperature than the atmosphere, the cold air in the sewers rises through the open gratings; hence the various systems for deodorizing the gases by the charcoal process. In the winter months our houses act as the furnaces; and as we try to make them as snug as possible, by reducing all in-draught through window fittings and doorways to the minimum, the open gratings, fitted to the main sewers, then act as feeders, sweeping a cold current of air through the sewers, which carries the foul gases along, and eventually rushes through faulty fittings into our comparatively warm dwellings. This must take place at all seasons of the year, and more especially during the night, when we cannot open our windows for ventilation, and when our rooms are at a higher temperature than the atmosphere. However, the more the sewers are properly aerated, reducing the in-draughts into our dwellings, from these noxious receptacles at all seasons of the year, the more healthful will our habitations become. So I am forced to condemn the plain "Roman" system, and advocate the suction fan method adopted by the modern miner.

The furnace system, for promoting ventilation, we all know to be a very expensive plan, unless we can utilize, as in large manufacturing districts, the numerous furnaces under steam boilers. This plan has, no doubt, certain advantages in being able to deodorize the gases, by passing them through the furnace, and then up tall chimneys, where the purified vapors would be wafted away into infinite space.

Next comes the suction fan, driven by the steam engine. Now we have a furnace, and at the same time we create a powerful current of air passing into, and drawn out of the sewers, and then through the furnace and up the chimney to the clouds.

With fans propelled by wind we have a power uncertain and capricious. However, I consider Archimedean screw ventilators could be applied with a measure of success in many small villages where economy may be a desideratum. They are very sensitive; the least wind will cause them to revolve, and with such an apparatus placed on the top of a high chimney, the revolving wheel being six feet in diameter, we should have an engine of considerable power. These machines are fitted with spiral screw blades for creating an upward current. At times the action is feeble; the gases, however, would be kept in motion. In high winds the revolutions are considerable, and the gases would be screwed out of the sewers more rapidly, and in-

stantly dispersed. Gas engines may be used for driving the fan, as some authorities may object to steam engines and smoke chimneys studded over large and fashionable localities; but when we consider that science has rendered those shafts smokeless, the engine becomes a matter of convenience, and a good gas engine of moderate power requires less skill on the part of the attendant.

Lastly, we have to consider the turbine or water engine for driving the fan. I have shown that, for the water-carriage system for the removal of refuse, more flushing water becomes imperative, and I propose to take advantage of the effluent water flowing from the turbine to effect a thorough cleansing of our underground netways. And, to satisfy the most fastidious, I propose drawing the gases through an inclosed bed of charcoal of sufficient capacity to suit the requirements.

I will now draw attention to these suction fans. A fan driven by an engine of 20 horse-power discharges 25,000 cubic feet per minute, the suction pipe being 30 inches in diameter, one mile of such pipe contains 25,872 cubic feet, or a little more than the above fan delivers per minute. The mileage per hour drawn through this 30-inch pipe will be 58, in round numbers, and in the 24 hours 1,392 miles of air are drawn through this pipe of 30 inches in diameter, while the total quantity of air expelled by the machine is 36,000,000 cubic feet in the 24 hours, or, in other words, thirty-six millions of cubic feet would be drawn out of a certain sewerage area, and pure air would instantly fill up the void, pouring into the sewers through existing apertures, or properly constructed inlets. To place these fans in immediate connection with the sewers is impracticable. As the numerous gully holes would act as feeders, and the exhaustive power, at a short distance, would become inoperative, we must lay down a system of piping, so that the sewers may be attacked, and the gases gently drawn out over a large area. Were we to create a sudden rush of air at one part of the sewerage system, even although with no gully holes, we consider the water traps in the house arrangements would be unsealed, and, as we have already stated, the exhaustive power would be limited.

For sake of illustration, I will take a straight length of piping, commencing with 30 inches diameter at the fan. This line of piping should be graduated to a small diameter at the extreme end of, say, one-sixth of the area, or 12¼ inches diameter, the end of the pipe being fitted with a blind flange. This being the main suction pipe, branches are cast on at certain intervals, and pipes fitted thereto in connection with the main sewer, or still smaller pipes in connection with each drain from the houses. These pipes should be smallest nearest the fan, and varying in diameter to the extreme end of the main suction pipe. We consider, that by this arrangement of graduated main and feeders, that the draught would be equalized throughout the entire length of pipe.

When it is desirable to lead a pipe from the main into each house drain, it must be placed between the sewer and the trap, fitted to the drain pipe. These feeders must be small in diameter, so that their combined area does not exceed the area of the suction pipe, as likewise to embrace as many house drains as practicable. For sake of illustration, we will take all the feeders of an uniform diameter of ½ an inch. The area of the 30 inches suction pipe is 706 square inches, and as the area of ½ an inch is .196, it will require 3,600 feeders to make up the area of the 30-inch suction pipe, or, in other words, 3,600 house drains will be in communication with the suction fan. As the fan discharges 25,000 cubic feet per minute, each feeder would be drawing from the sewers, say, 6.94 cubic feet per minute, or 416 cubic feet per hour. Thus it will be seen that the sewer air would be gently drawn out over the entire area to be ventilated.

We will now assume that a street contains 320 houses to the mile, the houses, of course, being on each side of the street. As there are 3,600 house drains to be ventilated, the total length of the street would be 11¼ miles. This shows the capability of a suction fan discharging 25,000 cubic feet of air per minute. We consider this arrangement of small feeders extends the area to be ventilated. Were the feeders one inch in diameter, more air would be drawn through each,

but there would be only 900 house drains ventilated, and the mileage would be reduced in proportion. In another arrangement, larger feeders are fitted to the main suction pipe, and placed in communication with the main sewers. These pipes would be placed further apart than the former arrangement, and graduated from a small diameter at the end nearest the fan, and increasing in diameter towards the extreme end, thereby tending to equalize the draught throughout the entire range of piping.

In some situations the main sewers may be of sufficient capacity, so that the suction pipe could be hung from the roof. It should be pierced with holes at certain intervals, which would act as feeders, drawing off the gases immediately over the parts when generated, and we consider all new works should be so arranged when practicable.

In the meantime we have to deal with existing arrangements. In some towns, steam-power would be preferable for driving the fan, and in other towns water pressure may be adopted. In laying down this proposed pneumatic mode of ventilation, central situations must be chosen. The area to be ventilated we will assume to be four square miles. A powerful suction fan must be erected centrally in that area, and placed in connection with a circular suction box, from which the various suction pipes would diverge. These pipes would be arranged through the main streets and cross streets, in a similar arrangement as for the distribution of gas for lighting purposes, and the feeders carried into the house drains, or in direct communication with the sewers, in the same way as the small gas pipes are taken into our houses. The one system of piping is identical with that of the other; the gas from the gas works is delivered under pressure; the pneumatic method of ventilation supplies air to our sewerage systems, by the exhaustion of a large volume of air drawn out of a system of piping, which, if properly arranged, cannot create negative pressure in the sewers, and which, we consider, would prove a great blessing to the community at large, by the thorough ventilation and purified dispersion of sewer gas carried up high chimneys, far overhead.

THE CONSOLIDATION OF FLUID STEEL.*

By ALFRED DAVIS, Westminster.

From "Engineering."

The difficulty of obtaining solid ingots under the ordinary system of casting in the Bessemer and Siemens process has induced several methods of applying pressure to the metal whilst in a liquid state.

Sir Henry Bessemer in 1856 took out a patent, under the title of "Manufacture of Iron and Steel," in which he proposes to use a hydraulic press as a means of condensing the ingot whilst in a semi-fluid state, and for which purpose he states a strong slide or cover must be made to close the mouth of the mould during the process.

The plan adopted by Sir Joseph Whitworth has been for some time in successful operation, and the process of hydraulic compression has also been practiced at the works of Messrs. Revollier, Bictrix, and Co., France, and at the Neuberg Works in Austria.

Mr. R. N. Daelen, of Barop, has a plan for pumping the fluid steel into a closed ingot mould. The three last systems are fully illustrated and described in the pages of *Engineering*, August 6, 1875, and October 8 of the same year, and are probably well known to the members of this Institute.

The system which the author proposes to describe is the invention of Mr. H. R. Jones, of the Edgar Thomson Steel Company, U. S. A., and is now in constant operation at the works of that company near Pittsburgh.

* Paper read before the Iron and Steel Institute at Liverpool.

The process is a very inexpensive one, and consists in simply admitting steam at a high pressure to the top of the ingot mould immediately after the metal has been poured.

A steam drum, or receiver, communicating direct with the boiler, is fixed, for the sake of convenience, to the side of the ingot crane. This drum has a number of cocks corresponding with the number of the moulds. India-rubber pipes are provided to conduct the steam, one end of the tube being permanently fixed to the drum, and the other by means of a coupling attached to the lid of the mould.

The accompanying drawings show the construction of the ingot moulds and attachments.

The base-plates upon which the stools rest are secured to a good foundation, and the stools are accurately fixed in position on the arc of a circle having the post of the ladle crane as a center. This is done to avoid racking in and out the ladle when pouring. The stools have projecting ribs to fit the base plates, and heavy lugs to which the moulds are clamped.

The ingot mould has at the upper end a cone seat accurately turned, upon which the pouring cup rests, and which afterwards receives the lid, which is secured in position by means of a steel wedge.

By this arrangement the cup is easily removed and the lid (with coupling and flexible pipe attached) substituted; the cone seat forming a steam-tight joint.

In coupling up the steam pipe, a shorter time is occupied than by the old method of filling up with sand.

For generating steam a cylindrical boiler is used 30 in. in diameter by 20 ft. long, and constructed to carry a pressure of 250 lbs. per square inch, although in practice a greater pressure than from 80 lbs. to 150 lbs. does not appear to be necessary, the higher pressure being used for mild steels.

The result obtained by the application of this process at the Edgar Thomson Works has proved completely successful.

Formerly at these works, with a 14 in. ingot reduced to a bloom of $7\frac{1}{4}$ in. $+ 7\frac{1}{4}$ in. it was necessary to cut off from 30 in. to 36 in. of the bloom in order to arrive at a part free from piping, whilst under this process the ingots are free from porosity, and are turned out with a perfectly level top.

A careful series of experiments has been made in order to ascertain the difference between an ingot cast in the ordinary way and one under steam pressure; and it has been found that the latter with the same quantity of metal from the ladle is from $1\frac{1}{2}$ in. to 2 in. shorter than the former when cold.

In the year 1878, when this process was first adopted, a saving of 2.6 per cent. was effected over the proceeding when the old method was in use; that is to say, what in the year 1877 was scrap, in the year 1878 was sound steel.

In addition to the consolidation of the ingot, there are several other advantages in this system. It is found that the steam, acting upon the end, cools and hermetically seals the top of the ingot, and enables the men to deal with it ten minutes earlier, without any fear of bleeding; and this allows the ingot to be conveyed to the reheating furnace with greater rapidity and in a hotter condition than formerly.

It is also found that with the use of steam the ingot moulds last better, the average in 1879 being 95 ingots, or nearly 112 tons of steel per mould.

In a paper read by Sir Joseph Whitworth at the meeting of the Mechanical Engineers in Manchester, July, 1875 several theories were discussed, as to the effect of compression on fluid steel, and although the result was generally admitted to be satisfactory, the subject was dismissed before any solution had been arrived at as to the *modus operandi*. During the discussion several speakers were of opinion that the gases were not forced out, but merely compressed, and consequently occupied so small a space that they could not afterwards be detected.

Mr. Daniel Adamson considered that the soundness of the compressed ingots could be accounted for in the following manner: That the metal running into the mould necessarily became cooled and solidified on its outer surface first, and that the natural contraction of the interior afterwards becoming cool must leave vacuous pores if allowed to cool in the ordinary way; but that the compression taking place during the time that the outer surface was becoming solidified, the metal

was welded together particle for particle, and the vacuous spaces avoided.

Sir Joseph Whitworth did not offer any explanation in regard to the expulsion of the gases under his system of compression, and it would be interesting to know the views he entertains on the subject.

Dr. Siemens observed that the result might be accounted for by the circumstance that the fluid steel, congealing first on the outside of the mould, offered more resistance there to the motion of the plunger, and the outside becoming thus, comparatively speaking, porous while the fluid portion in the center received a larger amount of compression than the outside, which had more power of resisting the pressure. The particles of gas entangled within the fluid mass would therefore encounter rather less resistance towards the outside than towards the inside, the full hydraulic pressure being transmitted to the center of the fluid mass, and in that way the expulsion of the gases from the fluid metal might perhaps be accounted for.

The following explanation from Mr. F. Moro, chief engineer of the Kladno Steel Works, Austria, in reference to the consolidation of ingots compressed under the steam process, appeared in a recent number of a scientific journal:

"The liquid steel has, like other metals, *e.g.*, lead, the peculiarity of absorbing gases, and the more of them the higher the temperature. On the other hand, these absorbed gases will come out of solution on cooling, accumulating in bubbles until the rigidity of the setting steel puts an end to this. This will be the case under ordinary atmospheric pressure, but it is otherwise when liquid steel is submitted to the influence of high pressure; then the absorbed gases remain in solution like carbonic acid in well-bottled soda-water, for example, and any formation of bubbles will become impossible at a pressure of over six atmospheres."

The method of steam compression described in this paper has recently been adopted at the works of Messrs. Bolckow, Vaughan, and Co., and although permanent arrangements have not yet been completed, sufficient has been accomplished to confirm the statements of our Transatlantic friends, and to justify the expectation that under this simple and inexpensive process, a result is gained equal to that obtained by the costly and elaborate systems of compression hitherto practiced.

THE DOUBLE-CHECK SYSTEM OF HOUSE DRAINAGE.*

By Mr. HENRY MASTERS, Architect.

From "The Architect."

Upon reviewing the various systems of house drainage that have been brought before the public during the past dozen years, I have been struck with the absence of simplicity in many of the schemes propounded. One school of sanitarians consider special-made traps essential. Another, the absolute necessity of mechanical contrivances for pumping up, or extracting the foul air from drains, though each agree that drain ventilation is necessary.

Nearly all the schemes put forward advocate up-currents of air by single pipes, and pumping terminations; some admitting air near the ground, and some not. If the multiplicity of special pipe terminations be evidence, down draft is a very formidable thing to contend with. In my scheme, I am a great friend to down draft; I advocate and encourage it in every possible manner, and the more I get of it the better. I, therefore, differ in this particular from other sanitarians.

In my early practice, as an architect and surveyor, the correct thing was to trap every closet, sink, and outlet; also, to put traps at the foot of every soil and rainwater pipe, and the overflows of tanks, the great object being to hermetically seal, so far as possible, every drain and soil-pipe; the latter was made tight at top and bottom, and the former at

* A Paper read at the Conference of the Society of Arts on Health and Sewage of Towns.

either end. The same principle of tight work was applied to our common sewers, and I regret that we have still advocates for sealing, both as regards the sewers of our towns, and also our private drains.

Upon applying my double-check system of house drainage, I always avoid using any speciality or mechanical contrivance. All the details required may be purchased at ordinary dealers in sanitary ware and ironmongers. I use generally the common form of trap, ordinary glazed ware pipes (sometimes with Sandford's joints), and stout cast-iron soil and air-pipes, selecting only careful and well-proved men for doing the work. My experience having taught me that simplicity is the best engineering, and the fewer parts the better, I decidedly object to any mechanical valves or revolving terminations.

As an example of simplicity of house drainage, and as a system which could never get out of repair, although we have, upon referring to the past, but few examples left us of the sanitary arrangements of our mediæval forefathers, I cannot refrain from bringing before you one such example of a well-arranged and ventilated system of domestic drainage, as applied to the Hospital of St. Cross, Hampshire.

The architect of St. Cross, in common with the forethought that governed our forefathers in the selection of suitable sites for their religious and domestic buildings, took care that abundance of water, and facilities for drainage, should be secured. In the case of St. Cross, the water from the River Itchin is higher above the hospital than below; advantage was taken of this circumstance to make a culvert from the higher part in the direction of the building, and in the garden was constructed an open pond, through which the water flowed, and was the source of domestic supply. It then passed on and round the Brethren's dwellings, and under a foot-bridge, constructed for the double purpose of communication with the garden, and for throwing over waste from the cooking department. The water passed on under several projecting wings of the building, in which were placed the closets, four in each wing, two up stairs and two down. In the domestic arrangements of St. Cross, each Brother had a complete suite of apartments. They were provided with a sitting-room and a bed-room, and also a scullery, where they washed, and from the sculleries were approached closets, one to each dwelling in the wings referred to, and here we have a perfect system of ventilated drains and open soil-pipes, the former an open channel formed by the river water being diverted, and the latter a shaft, about a yard square, extending from a short distance of the running water to the seat, by which the soil passed directly into the water, air being freely admitted to the soil-trunk, and passing out through the seats and open windows of the closets.

I admit that the system adopted by the Brethren at St. Cross would be hardly applicable to our modern views, but we can take a lesson from the works of these ancient men. Their sewers, as well as their soil-shafts, were open from end to end, and they avoided the complication of traps and cowls.

Unfortunately, in our cities, we are driven by circumstances to use close drains to some extent, and I hardly think we are prepared to advocate open channels for our sewage, through the middle of our streets, or open shafts for our closets; and, without expressing any opinion what others have done with special traps and cowls, I will describe a scheme I have applied during many years of my architectural practice, and in every case have found it effective.

The scheme I advocate for house drainage I have designated "The Double-Check System," because I apply a double trap and a double air-pipe, using three pipes altogether. Branches from the closets inside pass through the external walls, and join the soil-pipe, which is extended above the roof and clear of all windows; the curved termination of this pipe points in the direction of the house front. Where the drain passes out from the front of the house an air-pipe is connected; this also extends above the roof, the curved end pointing in the reverse direction to the soil-pipe termination.

Houses that are connected together are much more difficult to drain than when detached; my selected examples, as shown by the drawings exhibited, are

from houses in rows. If the houses are detached, I invariably put all drains outside the walls, and cut off all waste-pipes. I also ventilate all the drains as described.

The double-check system, in its most simple form of application, consists of a pipe drain from the back to the front of the house, if in a row, or outside the house if detached; at the back of the house it is joined to the vertical soil-pipe, to which is connected the w.c. pans. This soil-pipe continues its full size above the roof, and has a curved termination, pointing from the house (say to the right). At a point of the pipe drain or inside or in front of the house is connected a large iron pipe extending above the roof; this air-pipe has also a curved termination, the same as upon the soil-pipe extension, but it points in a reverse direction, or from the front of the house (say to the left); thus we have a U-shaped arrangement, an open pipe at the front of the house, with a left-hand curve, an open pipe at the back of the house with a right-hand curve, the lower ends of each pipe connected with the drain.

The foregoing explanations have reference to the house-drains proper, but unless accompanied by some arrangement for cutting off the public sewers from the house drains, the system would become a "public sewer ventilator," and to ventilate the public sewers at private expense, or to assist in establishing a current of sewer gas in the direction of the house, would be alike opposed to my views. I have, therefore, arranged a double trap for this purpose.

There is no specialty in this trap. It is simply two common siphon traps (as known in the building trade by that name), connected together by a short junction pipe, to which is attached a large iron gas exit or escape pipe, extending above the house roof. The action of this double-check sewer gas trap is as follows:—Upon such occasions as a pressure of gas present in the public sewers, capable of exerting sufficient force to pass a water trap, a portion of such gas will be forced into the house drain, and passing the first trap (which acts as a safety valve), will meet with resistance at the second trap, but, meeting with no further hindrance, will pass up the large gas-exit-pipe, and so escape into the air above the house roof. Sewer gas, therefore, either by being forced through the water of the first siphon trap, or being absorbed by the trap water, is prevented from passing into the house drains by a barrier, viz: the second siphon trap, which, in an occupied house, would always be charged with water. If pressure existed, it would be released as soon as the first trap is passed, because it would have a large pipe provided for its escape freely into the air, and above the roof.

The tops of all pipes are protected with terminations, the capacity of the air-ways being equal to the area of pipe. I object to any mechanical terminal, preferring only such contrivances that cannot get out of order, and that may be procured easily, and which will prevent birds making their nests in the pipes.

In order that any system of house drains may be perfect, it is absolutely necessary that facilities should be given, so that any accumulation of impure air in the drain or soil-pipes should be made to periodically "move on" and finally be expelled. All house drains and soil-pipes become somewhat foul in the course of time, and if I were to introduce an old soil-pipe into this room, you would, I am sure, object to be in its company. Many old soil-pipes have been found to be thickly lined with very disagreeable substances, giving off the most unpleasant scents, especially when in a damp state, and during hot, calm weather; and we all know the very unpleasant odor prevailing in many of our houses after the cooking of cabbage. These emanations should be got rid of as soon as possible.

In my experience of revolving cowls, which make one believe (during their revolutions) they are doing some good, I have found that at the time they are most required they are at a standstill, and act as obstructives, and when there is plenty of wind, and no necessity for screwing, they revolve at a furious rate. This must, of necessity, be the case during winds and calms, and in the curved termination I apply to my system I labor under the same disadvantage. When the winds blow strongly from nearly every point of the compass, a strong current of air passes into one of

my curved pipes through the drain and soil-pipe and out of the other, carrying with it any foul air that may be in the drain; but when there is a calm very little air is made to pass by natural currents. But, although I am anxious to take advantage of the passing currents of air, and they add to the value of my system by clearing the drains and pipes when the wind blows, I depend upon another power for obtaining the desired end.

In the present day, every closet is constructed so that it may be flushed by about two gallons of water; this quantity of water, if it were possible to be forced at once into the 4-inch soil-pipe, would form a solid piston of water, about a yard long, and would, in descending, push before it with considerable power the air in the pipes; and if the soil-pipe were 10 feet long, 10 feet of air would be displaced, and forced out of the open end of the pipe, at the same time air would pass into the top of the soil-pipe extension, and follow the water-piston, so that during the time it was acting as a displacer of foul air, it would be filling the vacuum with fresh air; and, if I had not such an objection to mechanical valves, I could easily contrive one that would retain the yard of water in the soil-pipe, which, being released suddenly, would act with very great power, but as in practice I have found it is desirable to lay on the water for pan-flushing by a large pipe, say, 1½-inch bore, taking care that the valve which allows the water to escape from the flush tank, and also the way into the closet-pan, be of equal capacity to the pipe, a good flush is procured, and the descent of water, passing down the soil-pipe, acting in a zig-zag and spiral form, produces an effect similar to the solid water-piston described, and it has the advantage of being more gradual in its action, and thereby displacing more foul air than if it acted suddenly. All open-top soil-pipes having no exit pipes, lose, to a large extent, this piston power for displacing foul air, and those having an opening near the ground (as in some systems) act as I have described, but with the disadvantage that, upon each occasion of the closet being used, a portion of foul air is forced out of such an opening, and it becomes in course of time very objectionable; for we all object to a disagreeable odor under our noses, and in the vicinity of our doors and windows. The double-check system has the advantage of being free from the objections I have pointed out, the soil-pipes and drains being open at either end, so that any foul air in them may be displaced, are easily acted upon, and I have found that a bucket of warm water thrown down the kitchen sink has caused a current of air to circulate through the pipes; and a leaky water tap, allowing a small quantity of water to pass down the drain constantly, has, during the night, and at other times, caused a moderate flow of air in one direction; and also in the arrangement of pipes, when one pipe is at the back of the house, and the other at the front, should the sun shine upon one pipe, the heat produced will cause a free circulation, so long as one pipe is warmer than the other. Of course, the various powers described do occasionally act one against the other; there may be times when the water-piston would be forcing air in one direction, and the bucket of water in the other, but in practice this has but little effect upon the successful working of the system. Upon examination, I have found the drain and soil-pipes very free from foul gases.

Some sanitarians advocate non-trapping, but I have a strong conviction of the usefulness of suitable traps for water-closets, baths, sinks, and lavatories.

The closet pans I prefer are of the wash-out type, with large flush pipes, thus avoiding D-traps and valves; and where practicable, I fix a simple form of grease and flush tank in a suitable position, so that the kitchen grease may be removed periodically, and buckets of water thrown in to flush the house drains at any time.

I have not attempted in my paper to bring before you the mode by which air and soil-pipes, when in sight, should be treated architecturally, but have preferred to confine my paper to a simple description of the double-check system; but I am quite alive to the fact that the treatment of soil and air-pipes should have some consideration from my professional brethren. There is no reason why they may not be successfuly treated

so as to harmonize with, rather than disgrace, the building with which they are associated.

The principles I have endeavored to establish are:—

1. That a mass of water thrown down a pipe will force air before it, and cause air to follow it, and thereby a temporary circulation of air will take place; and if the pipes are of the form of the letter U, air will pass into one leg and out of the other.

2. That a slight current of air, entering the bent ends of a U-shaped pipe, will cause air to circulate through such pipe.

3. That a flow of air through a water trap will pass through a second trap, unless there be means provided for its exit between the traps.

THE HOWE TRUSS BRIDGE QUESTION.

By C. L. CRANDALL, C. E.

Written for Van Nostrand's Magazine.

In the article on "A Wooden Howe Truss Bridge" published in the last Magazine, I find a few mistakes or oversights which, when corrected, considerably modify the results.

First. For the central truss the live load is assumed to be two locomotives weighing 2000 lbs. per foot on each track, and the web strains are found on the supposition that each train extends the whole length of the bridge.

With one track uniformly loaded, it is proved in elementary works on bridges that for any bay, the greatest main brace strain occurs when the bridge is loaded only to the remote end, and the greatest counter brace strain when it is loaded only to the near end. With two tracks and the trains moving in opposite directions, whenever the rear of one train meets the front of the other, both tracks will be loaded from this point to one end of the bridge, and unloaded from this point to the other end; and as this may happen at any point of the bridge, it follows that the web strains must be computed for the entire 2000 lbs. per foot, moving on from one end until covering the bridge. This gives (with the usual assumption, that the load is added as concentrated loads at each joint in succession) for the vertical components of the strains on the braces due to the moving load, 16500, 27500, 41250, 57750 and 77000 respectively, instead of 16500, 11000, 33000, 55000, and 77000 as given by the author. Hence, stress on first pair main braces

$$=(77000\times 1.13)+34230=121240;$$

on fifth pair, or counter brace,

$$=(16500\times 1.13)-4890=13755,$$

on the verticles,

$$5800+27500=33300,$$
$$10150+41250=51400,$$
$$18800+57750=76550,$$
$$27450+77000=104450,$$

respectively.

Second. The strains at the centers of the upper and lower chords are assumed equal. But at the upper chord the horizontal thrust of the central braces is taken up directly by the angle block, leaving the stress on this chord, even at the central point, less than on the lower one by the horizontal thrust of the central braces. The vertical component of the stress on the central braces, for uniform load over the entire bridge, equals $4328+11000=15328$.

Hence horizontal component

$$=15328\times\frac{11}{21}=8030;$$

and stress on 4th bay upper chord equals stress at middle lower chord, minus 8030,

$$=128460-8030=120430.$$

Again it is claimed that since the upper chord is composed of four pieces, each piece should be treated as a post acting separately. But these pieces are securely fastened together by bolts and packing blocks at two intermediate

points, leaving about 5 feet as the greatest length in which they can act separately. Each piece 5 feet long will resist, per square inch, by the formula used by the author

$$\frac{5000}{1+\frac{(5\times12)^2}{6^2}\times.004}=3570 \text{ lbs},$$

while for the whole beam, we have

$$\frac{5000}{1+\frac{(11+12)^2}{12^2}\times.004}=3370 \text{ lbs.},$$

which justifies taking the chord as a whole.

The angle and packing blocks reduce the section of the chord some 20 per cent., which with good joints should not materially reduce the strength for compression.

Hence breaking strength $=3370\times12\times24=970560$; or if we allow 10 per cent. for poor joints

$$=970560-97060=873500.$$

Third. The main and counter braces are securely bolted at their intersections. Hence only the free length, or about 11.5 feet, should be substituted in the formula, giving for the main braces,

$$\frac{5000}{1+\frac{(11.5\times12)^2}{8^2}\times.004}=2280 \text{ lbs. pr. sq. in.},$$

or $2280\times8\times10\times2=364800$ lbs. for breaking strength.

For the counters

$$-\frac{5000}{1+\frac{(11.5\times12)^2}{7.5^2}\times.004}=2120 \text{ lbs., and}$$

breaking strength

$$=2120\times7.5\times7.5=118720.$$

The vertical stress assumed by the author to come upon the counter is the portion of the weight upon the drivers, which, when at the third panel point, would tend to pass to the farther abutment, or $\frac{3}{8}$ of 44000. With the drivers of two engines meeting at the third panel point, since the driving wheel base is about 8 feet, only $\frac{7}{11}$ of the weight, or 28000 lbs. can be supported at the third panel point, while either the most of the 28000 lbs. on the trucks, or the 20000 to 40000 lbs. on the tenders must rest on panel points beyond the third, thus tending to relieve the counter more than the extra 6000 lbs. at the point tends to strain it, so that for this position of the two engines the stress on the counter must be less than that already found by considering the live load to move on to the bridge.

Fourth. The strength of the fourth bay of the lower chord is computed on the supposition of an actual cross-section of 13×24, while if of the usual type there are only 3 pieces and a splice plate about 1.5×13 at the center; and the angle blocks cut off 1 inch in depth, and about 3 in width at the panel point, leaving a section of 12×21.

When a pair of drivers from each engine rests on the center of the fourth panel, the rest of the bridge being loaded, we have a tension at the panel point of $\frac{128461}{12\times21}=510$ lbs. per square inch, due to the chord acting as a chord.

As a beam, it has a concentrated live load of 22000 lbs., and a distributed dead load of 4700 lbs. The effect of the track stringers will be to tend to distribute the concentrated load, and to carry a part, or say all, of the uniform load above them, viz: about 1700 lbs.

3000 lbs. distributed load is equivalent to $3000\times\frac{2}{3}=2000$ lbs. concentrated at the center.

Substituting in the formula for a beam built in at both ends and loaded at the center,

$$\frac{24000\times11\times12}{8}=\frac{R\times21\times12^2}{6},\ R=790,=$$

tension on upper fibres at panel point. Adding this to the tension existing as a chord, we have for the total tension,

$$790+510=1300 \text{ lbs.}$$

The breaking strength used by the author $=6000$ lbs. Hence factor of safety $=\frac{6000}{1300}=5$.

Collecting the results, we have

	Stress.	Breaking Load.	Factor.
1st Rod.....	104450	282000	2 7
2nd Rod.....	76550	212040	2.7
3rd Rod.....	51400	180000	3.5
4th Rod.....	33300	154800	4
Upper Chord..	120430	873500	7
End Braces...	121240	364800	3
Counter Brace.	13755	118720	8.5
Lower Chord..	——	——	5

None of the details of construction were given, hence the bridge was assumed to be of the ordinary type of Howe truss.

It may be worth while to note that should we use for the compression pieces the table given by Whipple,* an excellent authority on wooden bridges, the factor for the upper chord would be 7.5, and for the end brace 4.5, leaving the woodwork of the bridge quite well proportioned, although too light for the traffic.

* Practical Bridge Construction, p. 276.

RENDERING WOOD FIREPROOF.

From "The Building News."

At the meeting of the British Association recently held at Sheffield, amongst the many novelties that came before that body was one relating to the preserving of wood from fire. The inventor, according to the *Timber Trades Journal*, is Colonel P. P. De la Sala, and amongst the various models and specimens exhibited by him at the Cutlers' Hall, Sheffield, before the Association, were non-inflammable wood for building purposes, pliable wood, fretwork, besides shavings of non-inflammable wood for the manufacture of mats, rope, &c.

The groundwork of Colonel De la Sala's invention consists of, if not absolutely destroying, at any rate modifying the brittle and inflammable properties of dry vegetable substances, either in their compact or fibrous state. The inventor thus explains his peculiar method, already patented:—

"Though all alkaline compounds reveal the property of rendering vegetable matter more or less pliable and non-inflammable, I preferably make use of carbonates of fixed alkalies in the following way. I dissolve in cold or warm clear water carbonates of potash or soda, or I make use of them in a solution of filtered water heated to the boiling point, and add hydrate of lime to this solution, graduating the strength so as not to exceed a specific gravity of 1.060—if potash is used, or 1.050 when soda is used. In the first case the strength of the solution corresponds to about thirty grains of hydrate of potash to the fluid ounce, or about twenty grains of hydrate of soda to the fluid ounce. Wood to be used in naval construction, and in buildings or structures of wood on land, as well as vessels and land buildings already constructed, can be rendered fireproof by saturating the floors and decks and all exposed woodwork with alkaline lyes, and when dry the wood may be whitewashed, painted, or varnished in the usual way."

With reference to larger timber the inventor goes on further to state:—

"For boards, planks, or thicker pieces of timber I graduate the time of immersion so as to form a coating of from one-sixteenth to one-eighth of an inch, which can be obtained in from four to twelve hours, according to the more or less porous nature of the wood or the compactness of its fibre. I consider a coating of about one-eighth of an inch deep to be a sufficient fire protection for all kinds of timber for building purposes, as the spread of fire and great conflagrations generally originate in relatively small causes, such as burning cinders, dropped sparks from fireplaces, matches accidentally ignited, inflamed liquids, candles left burning; but the fireproof coating can be made deeper, or even go through the whole timber, in the event of its being considered desirable to combine great flexibility with absolute non-inflammability. In this case I make use of hy-

draulic or other pressure, so as to force the alkaline lyes through the wood to the extent desired."

When veneers are operated upon, it seems they are immediately afterwards rolled between steel rollers or pressed between steel plates. Wood thus treated, says Colonel De la Sala, is proof *against any ordinary heat.* Whether the invention will really accomplish as much as the originator claims for it, experience only will show. There may be considerable value in this process for destroying combustibility, but it remains to be seen whether it is better than every other mode of treating wood for the same objects, *and at the same cost.* Common paint or white lead thickly laid on renders wood to a certain extent unignitable from the painted surface, but if once the body of the wood is ignited the external paint checks the burning but little.

ELECTRICITY AS A MOTIVE POWER.*

By Prof. W. E. AYRTON.

From "Nature."

The lecturer commenced by referring to the stagnation of trade, to the various remedies that had been proposed to relieve it, and to the fact that while some were maintaining and others stoutly denying that commercial depression could be cured by legislation, we were too apt to forget that there existed a means by which, without lessening the wages of the workman or the profit of the master, the cost of production could be diminished, prices lowered, and the falling trade of England resuscitated. He next considered the consumption of coal for various purposes yearly in Sheffield, and showed that, although the price of coal in that town was very low, being only five shillings per ton for steam coal, the total annual cost for Sheffield alone must be something like £790,000. Actual instances were then given of great saving being effected by water-power being employed on a large scale for doing mechanical work. Contrasted with this, calculation showed that at the Falls of Niagara as much power was wasted as could be produced by the total present annual consumption of coal throughout the whole world. And when it was remembered that there existed in the world other waterfalls besides Niagara, that we had also innumerable rapidly-flowing rivers, the important fact, well-known to scientific men, but one which it was so difficult to induce the world at large to grasp, stared us in the face—that we obtained in a laborious way from the depths of the earth the power we employed, and we let run to waste, every hour of our lives, many, many times as much as we used.

Again, even in a perfectly flat country, where waterfalls were unknown, the question of the economic transmission of energy had no less interest; for large steam-engines could be worked much more economically than small ones, large steam-engines requiring a consumption of only two, or two and a half, pounds, of coal per horse-power per hour, whereas small steam-engines burned eight or ten pounds, or even more. And even where large economical engines were employed there was often, as in the gigantic cotton-spinning mills in Manchester, an enormous waste of power in the shafting used in transmitting it from the engine at the base of the factory to all the different floors, and parts of each floor, a waste so great that, in spite of the extreme inefficiency of small engines, it had been proposed, as an economical measure, to replace the one large steam-engine by many small ones, each driving two or three machines direct.

The lecturer then proved numerically that (contrary to the views expressed by some people) it was impossible to use economically in a town, for motive power, the water already brought in pipes to the houses for drinking purposes, since in most towns power so produced would cost about one shilling

* Abstract (by the Author) of the British Association lecture delivered to 4,000 workingmen of Sheffield, August, 23, 1879.

per horse-power per hour, and although in Sheffield the great head of the water would diminish this to about fivepence per horse-power per hour, still this had to be compared with considerably less than one farthing per hour, the low cost in Sheffield of producing each horse-power with a very *large good* steam-engine.

Experience was leading us to see that it was to electricity that we must resort to obtain a carrier that would, at a small cost, transport our motive power over long distances, and as an illustration that the electric transference of energy on a large enough scale to be of practicable value was possible, knives were ground on the platform by power conveyed about a quarter of a mile through wires carried over the houses, a Siemens' dynamo-machine being employed at the one end to convert into electricity the motive power supplied by a steam-engine, and a similar, but smaller, Siemens' dynamo-machine being used on the platform at the other end to reconvert into motive power the electric current conveyed by the wires.

The principles on which dynamo-electric machines and electro-motors act were then entered into fully experimentally, and reference was made to the first electro-motor ever made—that constructed by Salvator del Negro in 1831—as well as to the improvements introduced into it by Jacobi, who replaced the oscillating motion by a rotatory one; different forms of modern electro-motors were then shown in action driving sewing-machines, &c. It was mentioned that although Jacobi abandoned his electro-motor used to propel a boat on the Neva because the fuel cost too much, still, that the subject of electro-motors was none the less practically important because we had since learnt why the old form was such an expensive producer of power, and what was the proper duty to be performed by electro-motors.

It was this very question:—Can an electric engine be made to work more economically than a steam engine? that first attracted Joule, of Manchester, in 1843, to commence that all-important investigation, which lasted for six years, the determination of the mechanical equivalent of heat.

Formerly, electric currents were almost entirely produced by galvanic batteries, in which zinc was burnt just as in the furnace of a steam engine coke was burnt. The amount of heat that could be got from burning a pound of zinc could be ascertained in the same way as the amount of heat produced by the burning of a pound of coal, but the fact that the latter was about seven times the former was of little value in the science of electro-motors, until Joule had proved that a certain quantity of heat was always equivalent to exactly the same quantity of work, no matter how the heat be produced; had proved in fact that energy was as indestructible as matter, a law which had for one of its proofs the long unsuccessful search for a perpetual motion.

As a result of this law of the conservation of energy Prof. Ayrton went on to show that since a pound of ordinary coal burning gave out seven times as much heat as the burning of a pound of zinc, we might say at once, that a steam engine would give seven times as much work as an electric engine for equal weights consumed, if in both cases all the heat could be turned into work; or, since zinc was about twenty-four times as dear as coal, that a steam engine would be about 150 times as economical as an electro-motor, worked by a battery, if in both cases all the heat were converted into work.

But so far, he said, "we have only considered the law of 'Conservation of Energy.' There is, however, another, and no less important, principle called the 'Dissipation of Energy;' and this law tells us that although the energy of a system cannot by itself increase or diminish, yet our power to convert one form of energy into another is continually growing less, our stock of available energy is gradually failing. Our mountain lakes, our vast store of coal, are practically useless until either the water is set in motion rolling down the hillside, or until the particles of the coal are set in rapid vibration as it slowly burns; energy of position, energy of chemical affinity, are of no use to the manufacturer until turned into kinetic energy or energy of motion. But from friction of various kinds, whenever energy exists in the kinetic form, some portion of it is being continually converted into heat. Whenever man or nature utilizes energy, it must be first turned into some kinetic

form, and whatever be the aim of the special machinery employed, some of this energy passes into heat. We cannot make even a clock go without regular winding up, although the only useful work done by the clock is to turn its hands at regular speeds; the earth's energy of rotation is now, like the moon's in past ages, gradually growing less, and is being converted into heat on account of tidal retardation; the earth moon, and sun, and all the planets are losing their energies of motion and relative positions, to be all ultimately turned into heat.

"But at any rate, it will be said, there will still remain the heat, and since heat can be converted back into other forms of energy, we shall be none the worse off. But it must not be forgotten, that whenever heat is produced some passes off by conduction through even our best non-conducting substances, and by radiation into space from even our best non-radiating surfaces; and this conducted and radiated heat, although it may impart some trifling warmth to unseen worlds, is, for the greater part, entirely lost to our universe. And even were it not so, even had we perfectly non-conducting coatings, and perfectly non-radiating surfaces—had we, in fact, the most perfect heat engine that our study of the science of heat would lead us to believe theoretically possible, one with no friction, no loss of heat by conduction through the sides of our cylinders, and no radiation from their surfaces—still our power to convert heat into other forms of energy would be very limited. For if there are two bodies, one hotter than the other, we can employ an engine, like a steam-engine, to convert part of the heat in the hotter one into some other form of energy; but the amount of heat converted, with even this ideal perfect engine, will, with such temperatures as are met with in practice, only be a fraction of what necessarily passes through the engine from the hot body to the cold, and warms up the latter; and as our whole power of conversion of heat into work depends on the difference of temperature, we lose it altogether when we have brought all parts of a system to the same temperature, no matter how high this temperature may be.

"It is not, therefore, sufficient to say that the burning of a pound of coal produces seven times as much heat as the burning of a pound of zinc; but we must consider what fraction of the heat thus produced is converted into useful work in a heat and in an electric-engine respectively.

"As already mentioned, our most perfect steam-engines can be made to produce one horse power with the consumption of 2 lbs. of coal per hour. Now, the burning of 2 lbs. of coal will produce enough energy to raise 18,528,000 pounds one foot, or will produce 18,528,000 foot pounds of work. Now, one horse-power is equivalent to 1,980,000 foot pounds of work per hour; therefore, as regards the total energy in coal, even our best steam-engines only utilize $\frac{2}{9}$ of it, and waste $\frac{7}{9}$. But, as already mentioned, even a perfect engine cannot, with the ordinary temperature available, utilize the whole of the heat of the fuel. In fact, theory tells us that the efficiency of a perfect heat engine, or the ratio of the work done to the maximum work obtainable from the consumption of the fuel, is equal to the ratio of the number of degrees of temperature through which the steam is cooled in doing work to the highest temperature of our steam, when we take as our zero of temperature a point 460° below the ordinary zero of the Fahrenheit scale.

"Now, in our best steam-engines, the steam, when it begins to push the piston by expanding, has a temperature of about 300° F., and at the end of the stroke a temperature of 100° F., so that the efficiency of a perfect ideal engine working between these temperatures is only about $\frac{1}{4}$, not so very much greater than that of best practical engines.

"No great advance can be made, then, in a heat engine, except by making the temperature of the working substance, steam, gas, or whatever it may be, much higher. If, for example, we could raise the temperature of the working substance as high as, say, 3,000° F., the temperature of combustion, and could make it leave the engine without artificial cooling at the ordinary temperature of the air, which is, say, 60° F., then a perfect heat engine, under these conditions, would only waste about $\frac{1}{4}$ of the total energy; consequently, assuming that we

could, at these high temperatures, make a practical engine as good relatively to an ideal perfect engine as we can at lower temperatures, then a practical engine would only waste $\frac{1}{4\frac{7}{5}}$ of the total energy, or would have an efficiency of about 0·84.

"But, with our present knowledge, to work with steam or gas at a temperature of 3,000° F. is almost as ideal as an engine with no friction and with no loss of heat by conduction and radiation. We are, therefore, led to the conviction that as it is solely by working with steam at very high temperatures that the efficiency of steam-engines can be seriously increased, it may be well to consider whether it is not possible to economically replace the steam engine with some other form of motor."

It was then proved theoretically and experimentally that whenever an electro-motor is being worked by an electric current it is acting as a magneto-electric machine and producing a reverse current tending to stop the motion.

The lecturer then explained that, when an electro-motor is worked by a *given* galvanic battery, calculations lead us to the result that if we wish to produce the work *most economically* we must, by diminishing the load on the motor, allow its speed to increase until the reverse current it produces is only a little smaller than that sent by the battery; in fact, until the current circulating through the arrangement is very small, in which case the efficiency of the engine, or the ratio of the work it produces in a given time to the maximum work it could produce from the same consumption of material is nearly unity. If, on the other hand, we desire a *given* battery to cause the motor to do work *most quickly*, independently of the consumption of material, then calculation tells us that we ought to put such a load on the motor that its speed will send a reverse current equal to something like a half of the strength of the current the battery could send through the motor when at rest. In this case the efficiency is about $\frac{1}{2}$, or half the energy is wasted in heat.

He impressed upon the audience that the difference between these two considerations of maximum values ought carefully to be borne in mind, especially as it was usually the second—or how to obtain work *most quickly*—that had generally been taken into account, whereas it was the other one—or how to transmit work *most economically*, that would specially engage their attention during the lecture.

And in connection with the latter maximum value he said, "Let us consider that we work our motor in the most efficient way—that is very fast with a small load, and let us suppose as an extreme case that by so doing the efficiency is so little short of unity that we may regard it as one; then since an electro-motor worked by a battery in which zinc is burnt is 150 times as costly to maintain as a steam engine for equal efficiencies, the best electro-motor worked by such a battery will be thirty-three times as dear as our best steam engines having an efficiency of $\frac{2}{9}$. We may, therefore, throw on one side at once all idea of electro-motors worked by ordinary batteries even although the electro-motors be perfect. Now, this result is most important, since it shows—not that an electro-motor as a machine is inefficient, but it tells us that attempting to drive it with a galvanic battery is the hopelessly inefficient part of the arrangement.

"But if we turn to the question of using electro-motors for the transference of power, then there is no difficulty about burning zinc, and the high efficiency of such motors is all-important.

"For in the case of natural sources of power, such as waterfalls, we have merely to consider what amount of energy will be produced at the distant factory; will it be sufficent to repay the expense of putting up wires from the source to the factory, together with the cost of the two dynamo-electric ma chines, or will it be cheaper to put up and use a small steam-engine having probably an efficiency of only $\frac{1}{15}$?

"When the distance between the source and the motor is considerable, the cost of putting up the leading wires becomes important, and the question therefore arises, can two or more people use the same leading wires without increasing the thickness, or must the thickness of the wire be so much increased as to make the construction of two sets of leads as economical?"

Prof. Ayrton explained that he attach-

ed great importance to this question, because the answer to it would decide whether the electric transmission of power was a mere dreamer's fancy, or was likely to have a real commercial future.

A detailed examination was then made of the laws governing the transmission of energy by water power, and as a result of the fact that the energy of a flow of water depends on the quantity and on its head it was shown that, as far as the waste of power by friction of the water in the pipes was concerned, a great pressure in the reservoir sending a small current to turbines in a town also working at great pressure, was an extremely economical mode of transporting power, but that if we took into account the inefficiency of existing engines for producing a great pressure of water at the reservoir, combined with the great waste of power arising from even small leakages that were certain to be caused by the great water pressure, it followed that the system was an impracticable one.

An examination was then made of the laws governing the electric transport of energy, and the lecturer arrived at this result:

"Just as we concluded in the case of the water, that the most efficient method to employ in order to transfer the energy, was great pressure in the reservoir, combined with turbines in the town working at a high pressure, so now we conclude that the most efficient way to transfer energy electrically, is to use a generator producing a high electromotive force, and a motor producing a return high electromotive force; and by so doing, the waste of power in the transmission ought, I consider, be able to be diminished with our best existing dynamo-electric machines, to about 30 per cent.; for, as experiment shows the efficiency of our best existing dynamo-machines to be 0.86 (that is 86 per cent. of the power spent in revolving the bobbins is reproduced as an energy of electric current); therefore, if two similar dynamo-electric machines be coupled up to transmit power, and if they are worked most *economically* in the general way I have already explained, and with the details of arrangement that I will enter into later on, instead of being worked so that the motor gives out power *most rapidly*, I have reason for expecting that the combined efficiency of the arrangement can be made to closely approach the square of 0.86, and not merely one-half, as commonly supposed.

"But while the two solutions of the problem are thus identical, there is this most important difference; increasing the pressure of the water means an uneconomical task, while increasing the electromotive force set up by a dynamo-electric machine, or an electro-motor, means merely running it faster, or running it at the same speed and putting more wire on the rotatory portion.

"And again, assuming that the mean electromotive force between the wire and the earth be as much as one hundred times the elctromotive force producing the current, namely, the difference between the electromotive forces of the generator and of the motor, then with the ordinary insulation of the best land telegraph lines, less than one per cent. of the energy transmitted ten miles would be lost by leakage,

"It would be impossible to increase indefinitely the speed of revolution of the cylinder of an induction machine, since apart from mere mechanical friction the iron constituting the core of the revolving part has to be magnetised and demagnetised very rapidly as it revolves. Now, there is a physical limit to the speed with which this can be done, and in addition this rapid change of magnetism heats the iron very much. But experiment shows that at the ordinary speed of revolution of dynamo-electro machines, 700 turns per minute, the electromotive force is proportional to the speed. We are, therefore, very far yet from the limit of speed. Consequently it would be well for the transmission of power to attempt first, a considerable increase of speed in the generator, combined with so light a load on the motor, that its speed is also very high. When this begins to fail as larger and larger amounts of power are transmitted, then we might begin increasing the amount of wire on the revolving coils of each; but this, of course, has the objection that the loss of power from a given current would then become somewhat larger.

"In some of the dynamo-electric ma-

chines, the current that is sent through the external wires is the same as that which circulates round the fixed electro-magnets to create the magnetic field in which the movable coils revolve. Now, the small current which I am here advocating should pass between the generator at the one end of the line, and the electro-motor at the other, would be too small to properly magnetise the fixed electro-magnets of the two machines, so that even a high speed of the bobbin will not produce a high electromotive force. But this difficulty is easily overcome by the plan already employed, for totally different reasons, by Gramme, Lontin and Wilde, in their generator for producing currents for electric lighting, viz., that of using either a separate exciter, or a separate portion of the revolving bobbin in the generator, to produce the current to magnetise the fixed electro-magnets. In connection with this current for exciting the fixed magnets, it is worthy of notice, in passing, to observe that since experience shows that the electro-motive force of a dynamo-electric machine is proportional to the velocity, I conclude that the magnets are saturated, and that the exciting current is already too strong, so that it may be with advantage reduced, or many fewer coils of wire employed in this portion of the machine.

"We have then been led to this most important result, which I hope is clear to you all, and which I trust you may all carry away with you—*that a dynamo-electric machine, with a separate exciter, driven very fast with a steam engine, or with a stream of water, at high or low pressure, and sending, by even quite a fine wire, a small current to a distant electro-motor, also running very fast and magnetised by a separate exciter, is an economic arrangement for the transmission of power.*"

An examination was then made of the way this result was affected by increasing the length of the connecting wires, and it was proved that the electric transmission of power was not only practical, but also very economical, both for short and long distances, if the generator of the electric current at one end of the line and the motor, worked by this electric current, at the other end of the line, were both run fast enough, and if only we required to transmit a sufficiently large quantity of power.

The lecturer then went on to say, "We have been considering the transport of power derived more especially from natural sources; but since we have seen that by the use of electricity, properly employed, the waste of power in transmission can be reduced for any distance to about 30 per cent. of the whole power absorbed at the generator, it follows that the employment of steam-engines of vast size at points outside Sheffield would be by far the most economical mode of extracting the energy out of coal. For it is at least four times as expensive to produce power with a small steam-engine as with a large one; therefore, including the waste of power in electric transmission, the cost of production of power in small workshops would be little more than one-third as dear as if small steam-engines were used, and similarly the the waste of power in any large mill or factory in its transmission from the large steam-engine at its base to all the floors and machines on each floor would be very much diminished.

"Consequently it would be much more economical to work this lathe on the platform, as I will now proceed to do, by a big steam-engine in Howard street, several hundred yards away, than to use a small steam-engine here for this purpose."

He then reminded them that not only can electricity produce motive power, but also light and heat, and electric heating and lighting had this great advantage, that no chimneys were required. Experiments were then made of boiling water and lighting the Albert Hall by an electric current generated a quarter of a mile away.

Reference was then made to the great money saving of something like 30*s.* an hour, that Dr. Siemens had been able to effect at the Albert Hall, London, by replacing the old gas jets by electric lamps giving even more light, and to the unexpected advantage attained by the present stillness of the air arising from the use of the electric light, which enabled the singing and music there to be better heard now. Great weight was attached to the fact that at the Albert Hall the science of hanging a brilliant light high up had been luckily allowed to ride over

the precedent of putting a number of feeble glimmers all over a building, and in connection with this it was explained that the reason why electric lighting for streets had been economically much less successful, was because English conservatism had prevented the authorities from realizing the possibility of using for street electric illumination anything differing from an ordinary iron lamp-post. Attention was then drawn to the fact bearing most closely on the economy of electric lighting on a large scale, and which had been obtained as the result of experiments, that the larger were the dynamo-machines used for producing the electric light, the more light was produced per horse-power. Taking all this into consideration, Prof. Ayrton arrived at the result that "at any rate we may be absolutely safe in saying that the cost of using gas in Sheffield for lighting large halls, such as the one we are now in, factories and the streets could be halved if electric currents, generated by water engines worked by hill streams, as well as by very large steam engines, were substituted for gas.

"But, can this be quite right—for I have proved that to transfer energy economically we must use a large pressure and a small flow. Now, how can we produce a very bright electric light with a small current? Why, by not using the current that comes along the wire to produce the light at all, but merely to drive an electro-motor, which motor, at the place where any large amount of light was required, would be employed in giving motion to a second dynamo-electric machine, which would produce the currents for lighting purposes.

"This experiment I might show you, but as we have used already several times during the evening electric lights fed from a distance, we will vary the experiment and try an analogous one. Messrs Walker and Hall will now, at their works, give rapid motion to a dynamo-machine, and the current which, when properly arranged, as I have explained to you, may be small, will set in motion this electro-motor. This in its turn will cause this other dynamo-machine to rotate rapidly and produce a current which I will use for rapidly gilding this piece of plate."

Calculation showed that if electric currents generated by very large steam engines at certain points, and by turbines driven by the falling water on the hillsides round Sheffield, were substituted for the use of coal for motive power, smelting, heating, and lighting buildings, that a saving of something like £400,000 a year might be anticipated for that town; and as an argument to prove that although such a reform was startling in its economical bearing it might nevertheless be sound, the following was adduced:—Imagine the cost of cutlery and plated goods to remain as at present, but all machinery to be removed from Sheffield, then what an enormous loss would accrue to the town from everything having to be done by hand labor. The saving then which the lecturer was showing the audience how to obtain, enormous though it might be, was still small compared with the gain that the introduction of machinery during the last hundred years had effected for that town.

Next was considered whether the Sheffield Water Company had any water in their reservoirs that could be spared for producing motive power, since of course the water which did work at its source would lose head and so be unable to come to the tops of the houses in the town as at present, and it was shown that there was a considerable surplus supply. As an illustration of such a use of the water power, a two-inch board was sawn on the platform by a circular saw, driven by an electric current generated by a water engine in the yard of the Water Works, and conveyed to the Hall by wires crossing the streets.

As a practical illustration of what had been done the lecturer said:—"Last year two French engineers, MM. Chrétien and Felix, at Sermaize (Marne), actually ploughed fields by electricity, the electric current being produced by two dynamo-electric machines, of a form invented by M. Gramme, and shown in the diagrams on the walls. These machines were usually worked with a steam engine at some convenient place three or four hundred yards away in an adjoining road, and the electro-motors were also two Gramme machines, one on each side of the field, with their coils revolving, of course, backwards. Through one of these, the electric current was

sent alternately, so that motion was given to one or other of two large windlasses, one on each of the wagons containing the electro-motors. In this way the plough which could be used going in either direction, was first pulled across the field making a furrow, and then back again making another parallel furrow."

A photograph, taken on the spot, of one of the complete Gramme electro-motors, with its windlass and wagon, together with the double acting plough, was projected on to the screen.

A second photograph was also now projected on to the screen of M. Chrétien's electric crane for unloading boats. This too, the lecturer said, had been successfully employed for several months at Sermaize, in the harbor there, and it was considered that a saving of about thirty per cent. had been effected of the expense formerly incurred for unloading the sugar barrels out of the boats.

Reference was then made to the difficulty that would be experienced in distributing electric power properly, on account of the current, in any circuit being affected by any alteration in any other circuit connected with it, and it was explained how this difficulty was met by the electric current regulations of M. Hospitalier and Dr. Siemens. Another difficulty arising from the velocity of the water on the hill streams being great after floods, and small in dry weather, and which at first sight might appear to require an extravagant supply of dynamo-machines so that even in a draught sufficient power could be transmitted electrically, it was explained, could be overcome by storing up the electric energy as compressed gas, and it was shown that a square foot of hydrogen at thirty atmospheres pressure (the usual pressure in the iron gas bottles of commerce) combining with half a cubic foot of oxygen, at the same pressure, would develop no less than 110 million foot pounds of work.

Prof. Ayrton concluded by asking:—

"But is there no other side to this question? We are, it is true, a commercial people, but do we not still love our hills and our fields? There was a time when the cutler of now black, grimy, Sheffield was very fleet of foot in following the chase. There was a time, when, 'not only in the villages around old Sheffield,' so says the history of Hallamshire, 'were the file-makers' shops or the smithy to be seen, with the apprentices at work; but even on the hill side in the open country, at the end of the barn would be the cutler's shed, whilst in the valley below, by the river, was the grinding wheel, ready to sharpen the tools that had been manufactured.'

"And why not now? why should not that mountain air that has given you workmen of Hallamshire in past times your sinew, your independence of character, blow over your grindstone now? Why should not division of labor be carried to its end, and power brought to you instead of you to the power? Let us hope, then, that in the next century electricity may undo whatever harm steam may have done during the last, and that the future workman of Sheffield will, instead of breathing the necessarily impure air of crowded factories, find himself again on the hillside, but with electric energy laid on at his command."

Two different results which may arise when air which is saturated with moisture becomes chilled have been pointed out by M. G. Oltramare. The watery particles may collect in drops and fall by their own weight, leaving the air transparent; or they may remain in suspension and form a cloud. In the latter case he supposes that the molecules are kept apart by electrical repulsion. Jamin and other investigators have found that the watery particles in mists may remain liquid at fourteen degrees, or even at twenty degrees — 6.8 degrees or four degrees Fahrenheit—provided there is no sudden shock or solid contact, especially with ice. In such a condition of "superfusion," if there is a sudden discharge of electricity the separating force disappears, and the sudden union of particles, under their joint attractions, forms hailstones of various magnitudes. This explanation accounts for the rustling which is heard before a fall of rain, as well as for the accompanying electrical phenomena. The size of the stones depends partly on the thickness of the cloud, partly on the lowering of temperature in superfusion.

—*The Engineer.*

ELEMENTS OF THE MATHEMATICAL THEORY OF FLUID MOTION. PART II.

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IV.

§ 6.

ON THE FORM ASSUMED BY A ROTATING FLUID SPHERE ACTED ON BY THE ATTRACTIONS OF ITS OWN PARTICLES, AND BY THE FORCES GENERATED BY THE ROTATION.

The reader is supposed in this chapter to have a slight knowledge of the theory of attractions, sufficient at least to enable him to recognize the formulæ for the components of the forces acting on one body, due to the attraction of other bodies in space.

Let us assume the equation of an ellipsoid in the form

$$\frac{x^2}{c^2(1+\lambda'^2)}+\frac{y^2}{c^2(1+\lambda^2)}+\frac{z^2}{c^2}=1$$

Let X, Y, Z represent the components of the attraction of this ellipsoid upon a point at its surface. We have now, if ρ denote the uniform density of our homogeneous ellipsoid, and m denote the attraction between two units of mass at a distance of unity apart, for X, Y, Z the following known expressions:

$$X=-\frac{3Mmx}{c^3}\int_0^1\frac{t^2dt}{(1+\lambda^2t^2)^{\frac{1}{2}}(1+\lambda'^2t^2)^{\frac{3}{2}}},$$

$$Y=-\frac{3Mmy}{c^3}\int_0^1\frac{t^2dt}{(1+\lambda^2t^2)^{\frac{3}{2}}(1+\lambda'^2t^2)^{\frac{1}{2}}},$$

$$Z=-\frac{3Mmz}{c^3}\int_0^1\frac{t^2dt}{(1+\lambda^2t^2)^{\frac{1}{2}}(1+\lambda'^2t^2)^{\frac{1}{2}}}.$$

In which M denotes the mass of the body, and is equal to $\frac{4}{3}\rho\pi c^3(1+\lambda^2)^{\frac{1}{2}}(1+\lambda'^2)^{\frac{1}{2}}$. For the case of an oblate spheroid we of course have $\lambda=\lambda'$ and then by simple integration

$$X=-\tfrac{3}{2}\cdot\frac{Mmx}{\lambda^3c^3}\cdot\left(\tan^{-1}\lambda-\frac{\lambda}{1+\lambda^2}\right)$$

$$Y=-\tfrac{3}{2}\cdot\frac{Mmy}{\lambda^3c^3}\left(\tan^{-1}\lambda-\frac{\lambda}{1+\lambda^2}\right)$$

$$Z=-\tfrac{3}{2}\cdot\frac{Mmz}{\lambda^3c^3}(\lambda-\tan^{-1}\lambda)$$

We have also for the potential of a spheroid that differs but little from a sphere upon a point at its surface (vide Thomson & Tait Nat. Phil.)

$$V=4\pi am\left\{\frac{a}{3}+\Sigma\frac{S_i}{2i+1}\right\}$$

where S_i is a spherical surface harmonic.

Here we have of course

$$X=\frac{dV}{dx},\ Y=\frac{dV}{dy},\ Z=\frac{dV}{dz}.$$

Let now I denote the moment of inertia of the body with refereńce to the axis of z, ω its angular velocity about that axis, and μ its moment of rotation; we then hàve, as is well known,

$$I\omega=\mu;$$

also in the case of an ellipsoid, whose semi axes in the plane of X, Y are a, b, we have

$$I=\frac{M}{5}(a^2+b^2)=\frac{M}{5}c^2(2+\lambda^2+\lambda'^2)$$

$$=\frac{M}{5}\left(\frac{3M}{4\pi\rho}\right)^{\frac{2}{3}}\frac{2+\lambda^2+\lambda'^2}{(1+\lambda^2)^{\frac{1}{3}}(1+\lambda'^2)^{\frac{1}{3}}}$$

Leaving now these general considerations we will take up the more particular problem of the uniform rotation of a homogeneous mass of fluid, originally in a spherical form, and seek to discover what forms of equilibrium it may assume under the action of the generated forces. The axis of rotation is assumed to be z. In writing the differential equation of the fine surface of the fluid, there will necessarily enter the forces which have been instrumental in producing that surface, which of course differs from the fine surface of the fluid at rest. An integral of this equation can then not be obtained unless we know the values of these forces, but the forces in turn depend upon the nature of the surface; there is thus such a reciprocal relation existing between the forces and the surface, that a general solution of the proposed problem is impossible. We can however assume some

particular surface, and then determine what relations must exist between the various quantities in one differential equation and the variable parameters of the equation of this surface, in order that this equation may be an integral of the said differential equation. If such real conditions can be found, then the surface that we have assumed is one of the possible forms of equilibrium that our fluid mass may assume.

Let now X_τ, Y_τ, Z_τ, denote the components in the directions of x,y,z, respectively, of all the forces acting on any particle on the surface of fluid mass. For equilibrium we must have, as is well known,

$$X_\tau\, dx + Y_\tau\, dy + Z_\tau\, dz = o$$

We also have

$$X_\tau = X + f, \quad Y_\tau = Y + g, \quad Z_\tau = Z$$

when X, Y, Z, are the forces due to the attraction of the body upon the particle, and f and g are the components of centrifugal force. Reverting now to our previous equations giving X, Y, Z, we may write

$$X = -Px, \quad Y = -Qy, \quad Z = -Rz,$$

when the meaning of the abbreviations P,Q,R, is obvious. We have also, ω being the angular velocity of rotation,

$$f = \omega^2 x, \quad g = \omega^2 y.$$

The complete values of the forces under consideration are thus,

$$X_\tau = -(P-\omega^2)x$$
$$Y_\tau = -(Q-\omega^2)y$$
$$Z_\tau = -Rz$$

and the differential equation of the fine surface of the fluid is thus

$$(P-\omega^2)x\,dx + (Q-\omega^2)y\,dy + Rz\,dz = o$$

or,

$$\frac{P-\omega^2}{R}x\,dx + \frac{Q-\omega^2}{R}y\,dy + z\,dz = o$$

We wish now to determine what relations must exist between the coefficients in this differential equation and the coefficients entering into the differential equation of some surface, in order that this surface may be one of the forms of equilibrium of our rotating mass. Let us assume the ellipsoid as the proposed trial surface. We may write its equation in the form

$$\frac{x^2}{1+\lambda'^2} + \frac{y^2}{1+\lambda^2} + z^2 = c^2$$

This equation may represent ellipsoids and hyperboloids, according to the relative values and signs of λ and λ'; of course hyperboloids and paraboloids are not at all admissible forms. Differentiating this equation we have

$$\frac{x\,dx}{1+\lambda'^2} + \frac{y\,dy}{1+\lambda^2} + z\,dz = o$$

Now comparing this with the above differential equation of the free surface of the liquid, we see that in order that the ellipsoid may be a possible form of equilibrium, we must have

$$\frac{P-\omega^2}{R} = \frac{1}{1+\lambda'^2}, \quad \frac{Q-\omega^2}{R} = \frac{1}{1+\lambda^2}$$

Solving these for ω^2 we have,

$$(a)\begin{cases} \omega^2 = P - \dfrac{R}{1+\lambda^2} \\ \omega^2 = Q - \dfrac{R}{1+\lambda'^2} \end{cases}$$

Subtracting the second of these from the first

$$(1+\lambda^2)(1+\lambda'^2)(P-Q) - R(\lambda^2-\lambda'^2) = o$$

Substituting for P, Q, R their values, this becomes

$$(1+\lambda^2)(1+\lambda'^2)\int_0^1 \frac{[(1+\lambda^2t^2)-(1+\lambda'^2t^2)]t^2\,dt}{(1+\lambda^2t^2)^{\frac{3}{2}}(1+\lambda'^2t^2)^{\frac{3}{2}}}$$
$$-(\lambda^2-\lambda'^2)\int_0^1 \frac{t^2\,dt}{(1+\lambda^2t^2)^{\frac{1}{2}}(1+\lambda'^2t^2)^{\frac{1}{2}}} = o$$

or

$$(\lambda^2-\lambda'^2)\left\{(1+\lambda^2)(1+\lambda'^2)\int_0^1 \frac{t^4\,dt}{(1+\lambda^2t^2)^{\frac{3}{2}}(1+\lambda'^2t^2)^{\frac{3}{2}}} - \int_0^1 \frac{t^2\,dt}{(1+\lambda^2t^2)^{\frac{1}{2}}(1+\lambda'^2t^2)^{\frac{1}{2}}}\right\} = o$$

This equation may be satisfied in two ways; either by equating the quantity within the brackets to zero, or by making $\lambda^2-\lambda'^2=o$; this gives us the equation of one ellipsoid as,

$$\frac{z^2}{c^2}+\frac{x^2+y^2}{c^2(1+\lambda^2)}=1$$

which, λ being real, is the equation of an oblate ellipsoid. We have now to determine the values of λ which will be admissible in order that the fluid may be in equilibrium in the form given by this last equation. Either one of equations (a) serve for this determination. We will find, after the proper reductions, that λ is simply a function of the angular velocity of rotation, which indeed is obvious *a priori*. The second of equations (a) gives us

$$(1+\lambda^2)\ (Q-\omega^2)-R=o$$

giving Q and R their values this is

$$\frac{4\pi\rho}{2\lambda^3}(1+\lambda^2)[(1+\lambda^2)\tan^{-1}\lambda-\lambda]$$

$$-(1+\lambda^2)\omega^2-\frac{4\pi\rho}{\lambda^3}(1+\lambda^2)(\lambda-\tan^{-1}\lambda)=o;$$

for brevity write $\frac{\omega^2}{4\pi\rho}=S$; this equation then becomes

$$-\frac{3\lambda+2\lambda^3S}{2\lambda^3}+\frac{2}{2\lambda^3}\left(\frac{3+\lambda^2}{2}\right)\tan^{-1}\lambda=$$

or finally

$$\frac{3\lambda+2\lambda^3S}{3+\lambda^2}-\tan^{-1}\lambda=o$$

The velocity of rotation must be such that λ^2 shall be neither imaginary, equal to -1, or less than -1. If $\lambda^2=i\lambda'^2$ the surface is imaginary. If $\lambda^2=-1$ the surface is that of a paraboloid. If $\lambda^2<-1$ it is a hyperboloid; so that λ^2, if negative, must lie between o and -1. The way in which Laplace endeavored to determine the values of λ was to consider the first member of the above equation as the ordinate of a curve of which λ is the abscissa. Represent this quantity by ζ; then ζ is positive for $\lambda=o$ and $\lambda=\infty$. For $\lambda=o$ the curve will cut the axis of abscissas at the origin, and then the ordinates ζ will increase with λ until they obtain a maximum; they will then decrease until the curve again cuts the axis or until $\zeta=o$. The value of λ corresponding to this point will thus be a value corresponding to a form of equilibrium; the ordinates will now become negative, but as for $\lambda=\infty$ they are positive, the curve must again cut the axis, thus giving another new value of λ and a corresponding new form of equilibrium. For a maximum we have of course

$$\frac{d\zeta}{d\lambda}=o$$

or

$$(b).\quad \zeta\lambda^4+2(5\zeta-1)\lambda^2+9\zeta=o$$

This will have real and positive roots for $\zeta<\frac{1}{5}$; one of the roots of (b) will correspond to the maximum positive value of ζ, and another to the maximum negative value of this quantity, and from what we saw a moment ago it is evident that the greatest positive root of (b) corresponds to the negative maximum of ζ, and after passing this point the values of ζ will increase indefinitely. Eliminating ζ between (b) and $\zeta<o$ we have

$$\tan^{-1}\lambda-\frac{(7\lambda^2+9)\lambda}{(\lambda^2+1)(\lambda^2+9)}>o$$

The first member of this inequality and its derivatives are zero for $\lambda=o$; the derivative becomes negative as λ increases, becoming again zero for $\lambda=\sqrt{3}$, after which it remains constantly positive. Accordingly, the preceding inequality will be satisfied for all values of λ greater than the positive root of

$$\tan^{-1}\lambda-\frac{7\lambda^2+9}{(\lambda^2+1)(\lambda^2+9)}=o$$

An approximate value of this root is

$$\lambda=2.5293.$$

The corresponding value of S is

$$S=0.1123$$

If the above equation has a greater root than 2.5293 equation (b) will give us two values of λ corresponding to forms of equilibrium. Thus for

$$\lambda=2.5293 \text{ and } S=0.1123$$

there is but one form of equilibrium, but for

$$\lambda>2.5293, \text{ and } S<0.1123$$

there are two forms. As λ increases S decreases; this is easily seen for we have,

$$S=\frac{2\lambda^3}{(\lambda^2+1)(\lambda^2+9)}=\frac{2}{\left(1+\frac{1}{\lambda^2}\right)(\lambda^2+9)}$$

the denominator of this increases as λ increases, and consequently the value of the fraction diminishes.

Giving now to S its value we have,

$$\frac{\omega^2}{4\pi\rho}=0.1123$$

This gives the greatest angular velocity compatible with the ellipsoid of revolution, being one of the forms of equilibrium of one fluid mass. We saw that two forms of equilibrium arose by λ increasing. One of the values of λ [from 6] will increase indefinitely, thus giving an ellipsoid which becomes flatter and flatter until it becomes a mere elliptical disk; the other value of λ decreasing gives us in the limit of $s=o$ a sphere of radius c. Laplace has shown in the Mec. Cel. the impossibility of the Prolate ellipsoid existing as a form of eqilibrium by allowing λ^2 to become negative *i.e.*, $\lambda=\pm i\lambda'$. He shows that the value of λ so obtained will not vanish but remains imaginary; therefore there can be no prolate ellipsoid as a form of equilibrium. The preceding results have been obtained on the supposition that the angular velocity was given, the problem however admits of solution under another form, viz., having given the moment of momentum of the rotating body and its mass, we can determine the eccentricity of the corresponding ellipsoid. If a mass of fluid be left to itself in any state of motion it will preserve unchanged its moment of momentum, and its viscosity will, if the fluid remain continuous destroy all relative motion among the parts of the fluid so that it will ultimately rotate as a rigid body. If the final form be that of an ellipsoid of revolution there will be, as we shall see, but a single possible value of the eccentricity. But this form may become unstable if the moment of momentum exceed some limit depending on the mass of the fluid. Our preceding equations give us,

$$\frac{\omega^2}{4\pi\rho}=\frac{1}{2\lambda^3}[(3-\lambda^2)\tan^{-1}\lambda-3\lambda]$$

Since however $I\omega=\mu$ we have

$$\frac{25\mu^2}{6M^2}\left(\frac{4\pi\rho}{3M}\right)^{\frac{1}{3}}=(1+\lambda^2)^{\frac{2}{3}}\frac{(3+\lambda^2)\tan^{-1}\lambda-3\lambda}{\lambda^3}=q$$

in which q is known as a function of mass and moment of momentum. The second member of this is zero, for $\lambda=o$ and infinite for $\lambda=\infty$. Consequently there must be some real and positive value of λ which will satisfy the equation. In order to show that there is but one value of λ that will satisfy the equation it is necessary to show that the derivative of the second member with respect to λ is positive for every positive value of λ. Write for brevity

$$\frac{(3+\lambda^2)\lambda}{3+2\lambda^2}-\tan^{-1}\lambda=f(\lambda)$$

Then our derivative will be

$$\tfrac{1}{3}(1+\lambda^2)^{\frac{1}{3}}\left(\tan^{-1}\lambda+\frac{9(3+2\lambda^2)}{\lambda^3}f(\lambda)\right)=\frac{dq}{d\lambda}$$

The sign of this for a positive λ depends simply upon that of $f(\lambda)$. $f(\lambda)$ is zero for $\lambda=o$. We have now

$$f'(\lambda)=\frac{\lambda^2(1+2\lambda^2}{(1+\lambda^2)(3+2\lambda^2)^2};$$

this is positive for every positive value of λ therefore $f(\lambda)$ is itself always positive, and consequently

$\frac{dq}{d\lambda}$ will be always positive if λ is positive.

So then for each value of q or μ there is but one form of equilibrium.

The preceding remarks have all been made on the supposition that in the equation of our ellipsoid $\lambda=\lambda'$; this was a solution of a certain equation of condition that we were led to, viz.,

$$(\lambda^2-\lambda'^2)\left\{(1+\lambda^2)(1+\lambda'^2)\int_0^1\frac{t^2dt}{(1+\lambda^2t^2)^{\frac{3}{2}}(1+\lambda'^2t^2)^{\frac{1}{2}}}-\int_0^1\frac{t^2dt}{(1+\lambda^2t^2)^{\frac{1}{2}}(1+\lambda'^2t^2)^{\frac{3}{2}}}\right\}=o$$

This equation is however satisfied by equating to zero the factor within the parenthesis. This factor may be thrown into the form

$$\int_0^1\frac{t^2(1-t^2)(1-\lambda^2\lambda'^2t^2)dt}{(1+\lambda^2t^2)^{\frac{3}{2}}(1+\lambda'^2t^2)^{\frac{3}{2}}}=o$$

(see Résal *Mec. Cel.* p. 191). The sign of the integral depends on that of the factor $(1-\lambda^2\lambda'^2t^2)$. If the equation of the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

we know that,

$$\lambda^2=\frac{b^2-c^2}{c^2},\quad \lambda'^2=\frac{a^2-c^2}{c^2};$$

suppose these quantities to have opposite signs, *i.e.*, either b^2 or a^2 is less than c^2, or the axis of rotation is imtermediate in length between the greatest and least axis; but this supposition will cause the factor $(1-\lambda^2\lambda'^2t^2)$ to remain always positive, consequently the integral will be always positive. Now suppose λ^2 and λ'^2 both negative; this gives

$$c^2>a^2$$
$$c^2>b^2$$

Then from the above values of λ and λ' we see that they will always be less than unity, and so $\lambda^2\lambda'^2t^2$ will be positive and less than unity and the factor $(1-\lambda^2\lambda'^2t^2)$ will be always positive. Again; suppose λ^2 and λ'^2 both positive, then

$$c^2<a^2$$
$$c^2<b^2;$$

then it is clear that values of λ and λ' can be found that will make $(1-\lambda^2\lambda'^2t^2)$ either positive or negative, and consequently values can be found that will make the integral vanish. From this we deduce the theorem. If the ellipsoid of these unequal axes is a form of equilibrium, then the axis of rotation must be the least axis of the ellipsoid. If we have

$$\lambda^2<\frac{1}{\lambda'^2}$$

then $\lambda^2\lambda'^2<1$ and the factor $(1-\lambda'^2\lambda^2t^2)$ will be constantly positive, and the integral cannot vanish. Thus at least one of the quantities λ^2 or λ'^2 must be greater than unity, or, what is the same thing, one of the fractions $\frac{a}{c}$ or $\frac{b}{c}$ must be greater than $\sqrt{2}$ since $\frac{a}{c}=\sqrt{\lambda'^2+1}$.

Differentiating the equation of the ellipsoid we have

$$\frac{xdx}{a^2}+\frac{ydy}{b^2}+\frac{zdz}{c^2}=o$$

The differential equation of the free surface of the fluid is

$$-(P-\omega^2)xdx-(Q-\omega^2)ydy-Rzdz=o$$

and if this is to represent an ellipsoid, we must have the forces $-(P-\omega^2)x$, $-(Q-\omega^2)y$, $-Rz$ proportional to the quantities $\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}$, and the resultant of these forces will consequently be proportional to

$$\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}};$$

But this is the inverse length of the perpendicular from the center of the ellipsoid to the tangent plane, at the point under consideration. This theorem is due to M. Liouville. A discussion of the ellipsoid as a form of equilibrium is given by Mayer in Bd. 24 of *Crelle's Journal*. The subject was also taken up by M. Lionville and made the theme of a very elegant investigation in Tome 16 of the "*Journal de Mathematique*." The complete results of the investigation show that when $\frac{\omega^2}{4\pi\rho}$ is less than 0.09356, the surface of equilibrium may be one of the oblate spheroids before mentioned, or it may be the elliposid with three unequal axes. When $\frac{\omega^2}{4\pi\rho}=0.09536$ the last form no longer exists but passes into one of the preceding forms. For all values of $\frac{\omega^2}{4\pi\rho}$ between 0.09536 and 0.1123 ellipsoids of revolution alone exist, and at this limit they coincide; we have already seen that beyond this limit the ellpsoid cannot exist as a form of equilibrium.

The problem of finding the form of equilibrium which can be assumed by a homogeneous fluid mass, animated by a uniform motion of rotation when this mass differs in form but little from a sphere, has been made the subject of very rigorous investigation by Laplace and by numerous writers since his time. The results of these investigations show that the form is still in this case that of an oblate spheroid, of which the *applatissement* is equal to five-fourths of the ratio of the equatorial centrifugal force to the attraction at the surface. If ω^2c be the acceleration due to centrifugal force at the equator, and $\frac{4}{3}\pi c\rho$ the attraction, we have for the *applatissement*

$$\frac{5}{4}\cdot\frac{\omega^2}{\frac{4}{3}\pi\rho}=\frac{15}{16}\cdot\frac{\omega^2}{\pi\rho}$$

This form is evidently one which would result from a slow motion of rotation;

Indeed assuming that the angular velocity ω is small we could obtain this result from our previous investigations. It is quite possible that the fluid might have originally an angular velocity greater than that given by the equation

$$\frac{\omega^2}{4\pi\rho}=.1123$$

and that still it would find a form of equilibrium in one of the ellipsoids; because, as the angular velocity increases, the centifugal force increases, that the spheroid will tend to become flatter and flatter, but the flattening of the spheroid will tend to diminish the angular velocity, and a series of oscillations will ensue which the viscosity of the fluid will finally destroy, and thus the mass may arrive at such a condition as to enable it to assume and retain the form of an ellipsoid.

It may be of interest here, before leaving this problem, to give Sir Wm. Thomson's solution of the following problem:

A mass of homogeneous incompressible fluid of an originally spherical form is, by the action of a force distributed over its whole surface, slightly deformed; the force is then removed, and the spheorid left free to its own action; no forces now acting, other than the attractions among the particles, a series of oscillations will commence, required to find the time of oscillation. The deformation is supposed to be according to a spherical harmonic of order i. Assume h the height of the position of a particle on the deformed surface above its original position on the surface of the sphere. We can express h by the equation

$$h=S_1+S_2+\ldots S_i$$

when S_1, $S_2\ldots$ are spherical surface harmonics, and of course functions of the angular co-ordinates of the point. We know that the operator

$$\frac{x}{\gamma}\frac{d}{dx}+\frac{y}{\gamma}\frac{d}{dy}+\frac{z}{\gamma}\frac{d}{dz}$$

applied to any function, expresses the rates of variation of that function in the direction of the line through the region, and the point whose co-ordinates are x, y, z; applying it then to one function φ, we will obtain the variation of h, or the radial component of the velocity of a particle. This velocity will of course be represented by $\frac{dh}{dt}$, so we have the equation

$$\frac{dh}{dt}=\frac{x}{\gamma}\frac{d\varphi}{dx}+\frac{y}{\gamma}\frac{d\varphi}{dy}+\frac{z}{\gamma}\frac{d\varphi}{dz};$$

the function φ, however, satisfies the equation

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}=0;$$

therefore φ can be expanded in a series of spherical harmonics. Denoting then by a the original radius of the sphere, and by $\gamma(=a+h)$ this radius after reformation we have (*vide* Ferrers Spher. Har.)

$$\varphi=\Phi\left(\frac{\gamma}{a}\right)+\Phi_2\left(\frac{\gamma}{a}\right)^2+\ldots.\Phi i\left(\frac{\gamma}{a}\right)^i\ldots.;$$

also

$$\frac{dh}{dt}=\frac{dS_1}{dt}+\frac{dS_2}{dt}+\ldots\frac{dSi}{dt}$$

and

$$\frac{x}{\gamma}\frac{d\varphi}{dx}+\frac{y}{\gamma}\frac{d\varphi}{dy}+\frac{z}{\gamma}\frac{d\varphi}{dz}=\frac{1}{\gamma}\left\{\Phi_1\left(\frac{\gamma}{a}\right)+2\Phi_2\left(\frac{\gamma}{a}\right)^2+..i\Phi i\left(\frac{\gamma}{a}\right)^i\right\}$$

by Euler's theorum for homogeneous functions. From these last two equations we have by making $\gamma=a$, and since the terms having the same suffixes are homogeneous functions of the co-ordinates of the degrees 1, 2 . . . i respectively,

$$\frac{dS_1}{dt}=\frac{1}{a}\Phi_1,\ \frac{dS_2}{dt}=\frac{2}{a}\Phi_2,\ \ldots\ \frac{dSi}{dt}=\frac{i}{a}\Phi_i.$$

The expression which we have before given for the potential of the spheroid becomes, since $4\pi am=3g$,

$$V=ga+3g\Sigma\frac{S_i}{2i+1}$$

We have also for the fluid pressure the equation

$$\frac{p}{\rho}=V-\frac{d\varphi}{dt}-\frac{1}{2}\left\{\left(\frac{d\varphi}{dx}\right)^2+\left(\frac{d\varphi}{dy}\right)^2+\left(\frac{d\varphi}{dz}\right)^2\right\}+C.$$

But since the motions are infinitely small, the term

$$\left(\frac{d\varphi}{dx}\right)^2+\left(\frac{d\varphi}{dy}\right)^2+\left(\frac{d\varphi}{dz}\right)^2$$

may be discarded, and we have

$$C-p=\rho\left(\frac{d\varphi}{dt}-V\right).$$

The pressure at the free surface is a constant and may be denoted by π, then we have also for p the expression

$$p=\pi+\rho gh.$$

The variations of gravity involved in the infinitely small term may be disregarded. Eliminating p between these equations we have

$$C=\pi+\rho gh+\rho\left(\frac{d\varphi}{dt}-V\right);$$

Substituting for V its value

$$C_1=C_2+\frac{d\varphi}{dt}+gh-3g\left(\frac{S_1}{3}+\frac{S_2}{5}+\cdots\frac{S_i}{2i+1}\right)$$

when $C_2=\frac{\pi}{\rho}+gh$. Now differentiate this for t and we obtain, after obvious substitutions

$$o=\frac{d^2\varphi}{dt^2}+g\frac{dh}{dt}-3g\left(\frac{1}{3a}\Phi_1+\frac{2}{5a}\Phi_2+\cdots\frac{i}{(2i+1)}a\Phi_i\right)$$

Now in one expression for φ and $\frac{dh}{dt}$ make $\gamma=a$ and we have

$$\varphi=\Phi_1+\Phi_2+\cdots\Phi_i$$

$$\frac{dh}{dt}=\frac{1}{a}(\Phi_1+2\Phi_2+\cdots i\Phi_i)$$

These give us after substitution,

$$o=\frac{d^2\Phi_1}{dt^2}+\frac{d^2\Phi_2}{dt}+\cdots\frac{d^2\Phi i}{dt}+\frac{g}{a}[\Phi_1+2\Phi_2+\cdots i\Phi_i]-\frac{g}{a}\left\{\Phi_1+\tfrac{6}{5}\Phi_2\cdots\frac{3i}{2i+1}\Phi_i\right\}$$

This gives rise to the set of equations

$$\frac{d^2\Phi_1}{dt^2}+\frac{g}{a}\Phi_1-\frac{g}{a}\Phi_1=o$$

$$\frac{d^2\Phi_2}{dt^2}+\frac{g}{a}\frac{4}{5}\Phi^2=o$$

$$\cdot\quad\cdot\quad\cdot$$

$$\frac{d^2\Phi_i}{dt^2}+\frac{g}{a}\frac{2i(i-1)}{2i+1}\Phi_i=o$$

The integral of this last is known to be of the form

$$\Phi_i=A\cos\left\{t\sqrt{\frac{g}{a}\frac{2i(i-1)}{2i+1}}-E\right\}$$

When A is a surface spherical harmonic function of the co-ordinates of the point and is the maximum value of Φ_i, and E is the epoch of the simple harmonic function of the time. Using this solution in the equation $\frac{dS_i}{dt}=\frac{i}{a}\Phi_i$ we find for the time of oscillation,

$$t=2\pi\sqrt{\frac{a}{g}\frac{2i+1}{2i(i-1)}}$$

The oscillations will give rise to simple harmonic displacements of the normal displacement h. For the case of ellipsoidal deformation $i=2$ and we have

$$t=2\pi\sqrt{\frac{5a}{4g}}$$

Substituting in this for the constants these values, Sir Wm. Thomson finds for the half period

$$t=47^{m}\ 12^{sec.}$$

This is very nearly correct for the case of infinitely small motions which we have supposed. In the case of greater deformations we cannot obtain the time of oscillation, as there would exist non-periodic motions, and further, the term

$$\left(\frac{d\varphi}{dx}\right)^2+\left(\frac{d\varphi}{dy}\right)^2+\left(\frac{d\varphi}{dz}\right)^2$$

in the expression for p could not be disregarded.

§ 7.

MOTION OF A HOMOGENEOUS FLUID ELLIPSOID, WHOSE PARTICLES ARE ACTED UPON MERELY BY THEIR MUTUAL ATTRACTIONS.

The general solution of this problem has been given by Riemann and may be found either in his *Gesammelte Werke*, published by Dedekind and Weber, or in the 9Bd. *der Abhn. der Köng. Gessel, der Wissen. zur Gött.* 1861. In the previous year Dirichlet had published in the *Abhandlungen* a memoir upon the same subject in which he showed that possible motions of the fluid mass could be discovered by making the co-ordinates x,y,z, of any particle at the time t linear functions of the original values of these quantities, the coefficients being merely functions of this time.

Let the original line axes of the ellipsoid be denoted by a_o,b_o,c_o, at the time t these are a,b,c; similarly x,y,z, are the co-ordinates of any fluid particle at the time t the original values being x_o,y_o,z_o.

Now as was stated above we may represent the co-ordinates x,y,z, as linear functions of x_0,y_0,z_0, the coefficient being merely functions of the time, this we do by the equations

$$1.\quad \begin{aligned} x&=l\frac{x_0}{a_0}+m\frac{y_0}{b_0}+n\frac{z_0}{c_0},\\ y&=l'\frac{x_0}{a_0}+m'\frac{y_0}{b_0}+n'\frac{z_0}{c_0},\\ z&=l''\frac{x_0}{a_0}+m''\frac{y_0}{b_0}+n''\frac{z_0}{c_0}. \end{aligned}$$

Designate again through ξ, η, ζ, the co-ordinates of the point (x,y,z) referred to a system of axes that coincide at each instant with the principal axes of the ellipsoid, then $\alpha,\beta,\ldots\gamma''$ denoting the direction cosines we have,

$$2.\quad \begin{aligned} \xi&=\alpha x+\beta y+\gamma z\\ \eta&=\alpha' x+\beta' y+\gamma' z\\ \zeta&=\alpha'' x+\beta'' y+\gamma'' z \end{aligned}$$

And the known relations exist among the quantities $\alpha,\beta,\ldots\gamma''$ which enable us to write,

$$\xi^2+\eta^2+\zeta^2=x^2+y^2+z^2.$$

Again on account of the surface remaining always composed of the same particles we must have,

$$3.\quad \frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=\frac{x_0^2}{a_0^2}+\frac{y_0^2}{b_0^2}+\frac{z_0^2}{c^2}$$

Now divide each of the above equations for ξ,η,ζ, by a,b,c, respectively, and substitute for x,y,z, three values as given above in terms of x_0,y_0,z_0, then we shall clearly have the values of $\frac{\xi}{a}$, $\frac{\eta}{b}$ $\frac{\zeta}{c}$, in terms of $\frac{x_0}{a_0}$, $\frac{y_0}{b_0}$, $\frac{z_0}{c_0}$, and the coefficients will be new functions of the time, calling them $\alpha_1,\beta_1,\ldots\gamma_1''$ we have

$$4.\quad \begin{aligned} \frac{\xi}{a}&=\alpha_1\frac{x_0}{a_0}+\beta_1\frac{y_0}{b_0}+\gamma_1\frac{z_0}{c_0},\\ \frac{\eta}{b}&=\alpha_1'\frac{x_0}{a_0}+\beta_1'\frac{y_0}{b_0}+\gamma_1'\frac{z_0}{c_0},\\ \frac{\zeta}{c}&=\alpha_1''\frac{x_0}{a_0}+\beta_1''\frac{y_0}{b_0}+\gamma_1''\frac{z_0}{c_0}. \end{aligned}$$

Now, evidently again the quantities α_1, β_1,γ_1'' are the direction cosines of a new system of rectangular axis (by virtue of the preceding equation 3.), and can be called the direction cosines of a movable system ξ_1, η_1, ζ_1, with respect to the fixed system of x, y, z. Since we have

$$\begin{aligned} x&=a\xi+a'\eta+a''\zeta\\ y&=\beta\xi+\beta'\eta+\beta''\zeta\\ z&=\gamma\xi+\gamma'\eta+\gamma''\zeta \end{aligned}$$

and as above

$$\xi=a\alpha_1\frac{x_0}{a_0}\ldots$$

we find readily

$$\begin{aligned} x=&(aa\alpha_1+ba'\alpha_1'+ca''\alpha_1'')\frac{x_0}{a_0}\\ &+(aa\beta_1+ba\,\beta_1'+ca''\beta_1'')\frac{y_0}{b_0}\\ &+(aa\gamma_1+ba'\gamma_1'+c''\gamma a_1'')\frac{z_0}{c_0} \end{aligned}$$

and similar values for y and z; we have then by comparing these with the values given in the first equations for x, y, z,

$$5.\quad \begin{aligned} l&=aa\alpha_1+ba'\alpha_1'+ca''\alpha_1''\\ m&=aa\beta_1+ba'\beta_1'+ca''\beta_1''\\ n&=aa\gamma_1+b'a\gamma_1'+ca''\gamma.''\\ l'&=a\beta\alpha_1+b\beta'\alpha_1'+c\beta''\alpha_1''\\ m'&=a\beta\beta+b\beta'\beta_1'+c\beta''\beta_1''\\ n'&=a\beta\beta_1+b\beta''\gamma_1'+c\beta''\gamma_1''\\ l''&=a\gamma\alpha_1+b\gamma'\alpha_1'+c\gamma''\alpha_1''\\ m''&=a\gamma\beta_1+b\gamma'\beta_1'+c\gamma''\beta_1''\\ n''&=a\gamma\gamma_1+b\gamma'\gamma'+c\gamma''\gamma_1'' \end{aligned}$$

We have thus the means of determining the motion of the fluid particle, that is the values of l, $m\ldots n''$ at the time t in terms of the quantities a, b, c, and the positions of two systems of movable rectangular axis, viz: ξ,η,ζ, and ξ_1,η_1,ζ_1. Now in equations 11, of chapter I, make a, b, c equal to ξ, η, ζ respectively, and also observe that $\frac{dx}{d\xi}=a$, &c.,.. then these equations become

$$6.\quad \begin{aligned} \frac{d^2x}{dt^2}a+\frac{d^2y}{dt^2}\beta+\frac{d^2z}{dt^2}\gamma&=\frac{d}{d\xi}(\psi-\mathrm{P})\\ \frac{d^2x}{dt^2}a'+\frac{d^2y}{d^2t}\beta'+\frac{d^2z}{dt^2}\gamma'&=\frac{d}{d\eta}(\psi-\mathrm{P})\\ \frac{d^2x}{dt^2}a''+\frac{d^2y}{dt^2}\beta''+\frac{d^2z}{dt^2}\gamma''&=\frac{d}{d\zeta}(\psi-\mathrm{P}) \end{aligned}$$

this P might have been replaced by p, since the fluid is incompressible, and we may regard the density as equal to

unity. Now differentiate equations 2./ for t and for brevity, make

$$\frac{dx}{dt}\alpha+\frac{dy}{dt}\beta+\frac{dz}{dt}\gamma=\xi'$$

7. $$\frac{dx}{dt}\alpha'+\frac{dy}{dt}\beta'+\frac{dz}{dt}\gamma'=\eta'$$

$$\frac{dx}{dt}\alpha''+\frac{dy}{dt}\beta''+\frac{dz}{dt}\gamma'=\zeta'$$

then we have

$$\frac{d\xi}{dt}=\frac{d\alpha}{dt}x+\frac{d\beta}{dt}y+\frac{d\gamma}{dt}z+\xi'$$

$$\frac{d\eta}{dt}=\frac{d\alpha'}{dt}x+\frac{d\beta'}{dt}y+\frac{d\gamma'}{dt}z+\eta$$

$$\frac{d\zeta}{dt}=\frac{d\alpha''}{dt}x+\frac{d\beta''}{dt}y+\frac{d\alpha''}{dt}z+\zeta$$

and expressing again x, y, z in terms of ξ, η, ζ, these become

$$\frac{d\xi}{dt}=\left(\frac{d\alpha}{dt}\alpha+\frac{d\beta}{dt}\beta+\frac{d\gamma}{dt}\gamma\right)\xi$$
$$+\left(\frac{d\alpha}{dt}\alpha'+\frac{d\beta}{dt}\beta'+\frac{d\gamma}{dt}\gamma'\right)\eta$$
$$+\left(\frac{d\alpha}{dt}\alpha''+\frac{d\beta}{dt}\beta''+\frac{d\gamma}{dt}\gamma''\right)\zeta+\xi'$$

the expression for $\frac{d\eta}{dt}$, and $\frac{d\zeta}{dt}$ are obvious. Now differentiation of the known relations

$$\alpha^2+\beta^2+\gamma^2=1, \text{ \&c. . .}$$
$$\alpha\beta+\alpha'\beta'+\alpha''\beta''=h, \text{ \&c. . .}$$

gives

$$\alpha\frac{d\alpha}{dt}+\beta\frac{d\beta}{dt}+\gamma\frac{d\gamma}{dt}=o$$

$$\alpha'\frac{d\alpha'}{dt}+\beta'\frac{d\beta'}{dt}+\gamma'\frac{d\gamma}{dt}=o$$

$$\alpha''\frac{d\alpha'}{dt}+\beta''\frac{d\beta''}{dt}+\gamma''\frac{d\gamma''}{dt}=o$$

$$\alpha''\frac{d\alpha'}{dt}+\beta''\frac{d\beta'}{dt}+\gamma''\frac{d\gamma'}{dt}$$
$$=-\left(\alpha'\frac{d\alpha''}{dt}+\beta'\frac{d\beta''}{dt}+\gamma'\frac{d\gamma''}{dt}\right)=p$$

$$\alpha\frac{d\alpha''}{dt}+\beta\frac{d\beta''}{dt}+\gamma\frac{d\gamma'}{dt}$$
$$=-\left(\alpha''\frac{d\alpha}{dt}+\beta''\frac{d\beta}{dt}+\gamma''\frac{d\gamma}{dt}\right)=q$$

$$\alpha'\frac{d\alpha}{dt}+\beta'\frac{d\beta}{dt}+\gamma'\frac{d\gamma}{dt}$$
$$=-\left(\alpha\frac{d\alpha'}{dt}+\beta\frac{d\beta'}{dt}+\gamma\frac{d\gamma}{dt}\right)=r$$

and so we have

$$\xi'=\frac{d\xi}{dt}-r\eta+q\zeta$$

8. $$\eta'=\frac{d\eta}{dt}-p\zeta+r\xi$$

Now from equation 7. it is easy to get

$$\frac{d^2x}{dt^2}\alpha+\frac{d^2y}{dt^2}\beta+\frac{d^2z}{dt^2}\gamma=\frac{d\xi'}{dt}-r\eta'+q\zeta'$$

9. $$\frac{d^2x}{dt^2}\alpha'+\frac{d^2y}{dt^2}\beta'+\frac{d^2z}{dt^2}\gamma'=\frac{d\eta'}{dt}-p\zeta'+r\xi'$$

$$\frac{d^2x}{dt^2}\alpha''+\frac{d^2y}{dt^2}\beta''+\frac{d^2z}{dt^2}\gamma''=\frac{d\zeta'}{dt}-q\xi'-p\eta'$$

and, designating by p_1, q_1, r_1 the values of p, q, r when the quantities α, β. . . γ'' are replaced by α_1, β_1, . . . γ_1'', equations 4. give

$$\frac{d\frac{\xi}{a}}{dt}=r_1\frac{\eta}{b}-q_1\frac{\zeta}{c}$$

10 $$\frac{d\frac{\eta}{b}}{dt}=p_1\frac{\zeta}{c}-r_1\frac{\xi}{a}$$

$$\frac{d\frac{\zeta}{c}}{dt}=q_1\frac{\xi}{a}-p_1\frac{\eta}{b}$$

Writing these out in full and then substituting the values of $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$ in equations 8. these become

$$\zeta'=\frac{da}{dt}\frac{\xi}{a}+(ar_1-br)\frac{\eta}{b}+(cq-aq_1)\frac{\zeta}{c}$$

11. $$\eta'=(ar-br_1)\frac{\eta}{b}+\frac{db}{dt}\frac{\eta}{b}+(bp_1-cp)\frac{\zeta}{c}$$

$$\zeta'=(cq_1-aq)\frac{\xi}{a}+(bp-cp_1)\frac{\eta}{b}+\frac{dc}{dt}\frac{\zeta}{c}$$

The quantities ξ', η', ζ', are evidently the components of the velocity of the point (x,y,z) in the directions of ξ, η, ζ; $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$, are the components of the relative velocity, and p,q,r the components of angular rotation of the axis ξ, η, ζ; p,q,r have the same significance with respect to the system ξ,η,ζ.

Now let π denote the constant pressure at the free surface, then

$$p=\pi+\omega\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta}{c^2}\right)$$

Where ω is an unknown function of t. Also for the potential ψ,

$$\psi = H - A\xi^2 - B\eta^2 - C\zeta^2$$

Where H is a constant, and the forms of A,B,C are already known.

By virtue of the known relations between ξ,η,ζ, and x_o,y_o,z_o, we may write,

$$\psi = H - [Ax_o^2 + By_o^2 + Cz_o^2 + Dy_oz_o + Ez_ox_o + Fx_oy_o]$$

and likewise,

$$p = \pi - \omega\left(1 - \frac{x_o^2}{a_o^2} - \frac{y_o^2}{b_o^2} - \frac{z_o^2}{c_o^2}\right)$$

Now we have readily from equations 6. the following system of differential equations between the ten functions of the time, $l, m, \ldots . n''$ and ω

$$l\frac{d^2l}{dt^2} + l'\frac{d^2l'}{dt^2} + l''\frac{d^2l''}{dt^2} = -2a_oA + \frac{2\omega}{a_o^2}$$

$$l\frac{d^2m}{dt^2} + l'\frac{d^2m'}{dt^2} + l''\frac{d^2m''}{dt^2} = -2Fa_ob_o$$

$$l\frac{d^2n}{dt^2} + l'\frac{d^2n'}{dt^2} + l''\frac{d^2n''}{dt^2} = -2\varepsilon c_oa_o$$

12.
$$m\frac{d^2l}{dt^2} + m'\frac{d^2l'}{dt^2} + n''\frac{d^2l''}{dt^2} = -2Fa_ob_o$$

$$m\frac{d^2m}{dt^2} + m'\frac{d^2m'}{dt^2} + m''\frac{d^2m''}{dt^2} = -2Bb_o + \frac{2\omega}{b_o}$$

$$m\frac{d^2n}{dt^2} + m'\frac{d^2n'}{dt^2} + m''\frac{d^2n''}{dt^2} = -2Db_oc_o$$

$$n\frac{d^2l}{dt^2} + n'\frac{d^2l'}{dt^2} + n''\frac{d^2l''}{dt^2} = -2\varepsilon c_oa_o$$

$$n\frac{d^2m}{dt^2} + n'\frac{d^2m'}{dt^2} + n''\frac{d^2m''}{dt^2} = -2Db_oc_o$$

$$n\frac{d^2n}{dt^2} + n'\frac{d^2n'}{dt^2} + n''\frac{d^2n''}{dt^2} = -2Cc_o + \frac{2\omega}{c_o}$$

The incompressibility of the fluid furnishes another equation, viz:

$$abc = a_ob_oc_o$$

or as we have seen in the first chapter

12.
$$\begin{vmatrix} l, & m, & n, \\ l', & m', & n', \\ l'', & m'', & n'', \end{vmatrix} = 1$$

From these differential equations we obtain, as in Chapter I, the three following integrals:

$$m\frac{dn}{dt} - n\frac{dm}{dt} + n'\frac{dn'}{dt} - m'\frac{dm'}{dt} + m''\frac{dn''}{dt} - n''\frac{dm''}{dt} = \text{const.}$$

13.
$$n\frac{dl}{dt} - l\frac{dn}{dt} + n'\frac{dl'}{dt} - l'\frac{dn'}{dt} + n''\frac{dl''}{dt} - l''\frac{dn''}{dt} = \text{const.}$$

$$l\frac{dm}{dt} - m\frac{dl}{dt} - l'\frac{dm'}{dt} - m'\frac{dl'}{dt} + l''\frac{dm''}{dt} - m''\frac{dl''}{dt} = \text{const.}$$

In the case of a fluid as in the case of a system of points, D'Alembert's principle holds, and consequently the theorems of the Conservation of Energy, conservation of the motion of the center of gravity, and of the conservation of areas all hold. Thus we have the means of arriving at the remaining integrals. Obviously the principle of the conservation of the motion of the center of gravity can give nothing new as the equations must be identical, from the fact that we assume this motion to be zero.

Now let $d\tau$ represent an element of the fluid mass which for $t=o$ has the co-ordinates x_o,y_o,z_o then we have for the energy at the time t

$$T = \tfrac{1}{2}\int\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}d\tau$$

Now substitute here the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, from equations 1. Make $d\tau = dx_o\,dy_o\,dz_o$ and the variables of integration respectively $\frac{x_o}{a_o}, \frac{y_o}{b_o}, \frac{z_o}{c_o}$, and we readily obtain

$$\int x_o^2 d\tau = \frac{4\pi}{3}abc\frac{a^2}{5}, \quad \int y_oz_o d\tau = o$$

$$\int y_o^2 d\tau = \frac{4\pi}{3}abc\frac{b^2}{5}, \quad \int x_oz_o d\tau = o$$

$$\int z_o^2 d\tau = \frac{4\pi}{3}abc\frac{c^2}{5}, \quad \int x_oy_o d\tau = o$$

and so

14.
$$T = \frac{2\pi}{15}abc\left\{\begin{array}{l} a^2\left\{\left(\frac{dl}{dt}\right)^2 + \left(\frac{dl'}{dt}\right)^2 + \left(\frac{dl''}{dt}\right)^2\right\} \\ + b^2\left\{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dm'}{dt}\right)^2 + \left(\frac{dm''}{dt}\right)^2\right\} \\ + c^2\left\{\left(\frac{dn}{dt}\right)^2 + \left(\frac{dn'}{dt}\right)^2 + \left(\frac{dn''}{dt}\right)^2\right\} \end{array}\right\}$$

Further let U denote the potential of all the working forces—*i.e.*, in this case the potential of the ellipsoid upon itself

$$U=\int \psi d\tau$$

It is known to be equal to $\frac{4}{3}$ of the volume of the ellipsoid multiplied by the potential of the ellipsoid at its center, that is

15. $$U=\frac{16\pi}{15}a.b.c.H$$

Now the theorem of the conservation of energy gives us

16. $$T=U+\text{const.},$$

and the conservation of areas gives us these three integrals,

$$\int\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)d\tau,$$

$$\int\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)d\tau,$$

$$\int\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)d\tau.$$

equal to constants. These become

$$\text{const.}=a^2\left(l''\frac{dl''}{dt}-l''\frac{dl''}{dt}\right)+b^2\left(m'\frac{dm''}{dt}-m''\frac{dm'}{dt}\right)+c^2\left(n'\frac{dn''}{dt}-n''\frac{dn'}{dt}\right)$$

17. $$\text{const.}=a^2\left(l''\frac{dl}{dt}-l\frac{dl''}{dt}\right)+b^2\left(m''\frac{dm}{dt}-m\frac{dm''}{dt}\right)+c^2\left(n''\frac{dn}{dt}-n\frac{dn''}{dt}\right)$$

$$\text{const.}=a^2\left(l\frac{dl'}{dt}-l'\frac{dl}{dt}\right)+b^2\left(m\frac{dm'}{dt}-m'\frac{dm}{dt}\right)+c^2\left(n\frac{dn'}{dt}-n'\frac{dn}{dt}\right)$$

We have thus seven first integrals of our general differential equations. From this point we might take up several special cases, but though all are interesting we will only take up the one in which there is no motion of rotation or the directions of the principal axes remain unchanged during the motion. This supposition gives us,

$$x=l\frac{x_0}{a_0},\ y=m'\frac{y_0}{b_0},\ z=n''\frac{z_0}{c_0}$$

and $$l'=l''=m=m''=n=n'=o$$

It will make no difference in the nature of our results if we write $x=kx_0$, $y=n'y_0$, $z=n''z^0$, instead of the above forms, and by doing so we retain a similarity in our results to some previous ones, understanding that $l=\frac{l}{a_0}$, &c. We have now for ψ,

18. $$\psi=\pi a_0b_0c_0\int_0^\infty\frac{1-\frac{l^2x^2_0}{l^2a^2+\chi}-\frac{m''^2y^2_0}{m''^2b^2+\chi}-\frac{n'''^2z^2_0}{n'''^2c^2+\chi}}{\sqrt{(l^2a_0^2+\chi)(m''^2b_0^2+\chi)(n'''^2c_0^2+\chi)}}$$

Call the denominator under the integral sign D and our differential equations become

$$l\frac{d^2l}{dt^2}=\frac{2\omega}{a_0^2}-2\pi a_0b_0c_0\int^\infty\frac{d\chi}{D}\cdot\frac{l^2}{l^2a_0^2+\chi}$$

19. $$m'\frac{d^2m'}{dt^2}=\frac{2\omega}{b_0^2}-2\pi a_0b_0c_0\int^\infty\frac{d\chi}{D}\cdot\frac{m'^2}{m'^2b_0+\chi}$$

$$n''\frac{d^2n}{dt^2}=\frac{2\omega}{c^{02}}-2\pi a_0b_0c_0\int^\infty\frac{d\chi}{D}\cdot\frac{n''^2}{n''^2c_0^2+\chi}$$

and the determinant becomes

$$lm'n''=1$$

The equations 13. have their second members $=o$. This is the case—see first chapter—when the angular motion of the fluid particles is $=o$. And there exists in consequence a velocity potential. Equations 17. have their first members $=o$. And from the equation expressing the conservation of energy we can obtain a first integral.

Suppose that our ellipsoid is one of revolution around the axis of z, its initial equation will be

$$\frac{x^2_0+y^2_0}{a^2_0}+\frac{z^2_0}{c^2_0}=1$$

and at the time t

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1$$

The conditions that these shall be identical are easily found to be (under the new significance of $l, m, \ldots n''$)

$$l''=m''=n=n'=o$$
$$l=\pm m',\quad l'=\pm m$$

The ambiguity in the signs is removed by taking n'' as positive, then by aid of the determinant that we have used before we see that

$$l=m',\quad l'=-m$$

and the determinant becomes

$$(l^2+m^2)n''=1$$

and we find immediately

$$a^2=\frac{a_o^2}{n''},\quad c^2=c_o^2n''^2$$

and

$$x=lx_0+my_o,\ y=-mx_o+ly_o,\ z=n''z_o$$

These equations evidently represent a motion made up of certain displacements in the direction of the axes, together with a rotation around z. We have now for ψ,

$$\psi=\pi a^2c$$
$$\int_o^\infty n''\,\frac{1-\frac{x_o^2+y_o^2}{a_o^2+n''\chi}-\frac{n''^2z_o^2}{n''c_o^2+\chi}}{(a_o^2+n''\chi)\sqrt{c_o^2n''^2+\chi}}\,d\chi$$

and we obtain readily for our new differential equations

$$l\frac{d^2l}{dt^2}+m\frac{d^2m}{dt^2}=\frac{2\omega}{a_o^2}-2\pi a_o^2c_o\int_o^\infty\frac{n''d\chi}{(a_o^2+n''\chi)\sqrt{(n''^2c_o^2+\chi)}}$$
$$n''\frac{d^2n''}{dt^2}=\frac{2\omega}{c_o^2}-2\pi a_o^2c_o\int_o^\infty\frac{n''^2d\chi}{(a_o^2+n''\chi)(n''^2c_o^2+\chi)^{\frac{1}{2}}}$$
$$l\frac{d^2m}{dt^2}-m\frac{d^2l}{dt}=o$$
$$(l^2+m^2)n''=1$$

The third of these gives us the integral

$$ldm-mdl=c'dt$$

c' being a constant.

This is the only integral given by equations (13) and (17). The conservation of energy gives one more integral. The discussion of these results enables us to determine the motion (vibratory) of the surface of the ellipsoid.

BLAST-FURNACE SLAG.

From "Iron."

Scattered throughout the iron-making districts of Great Britain are many million tons of scoria or refuse from the blast furnaces, which is technically known as slag. This slag goes on accumulating at the rate of nearly eight millions of tons per annum, its bulk being some three times that of the iron from which it has been separated. It forms a heavy encumbrance to ironmasters, demanding the purchase of large tracts of land whereon to deposit it, the investment being, of course, wholly unremunerative. There are one or two exceptions to this rule, as at the Barrow Hematite Iron Works, where the slag is tipped into the sea and serves to form land for the works, and at Middlesbrough, where some of the iron works supply slag for the construction of the breakwater and training walls in the river Tees. The quantity thus utilized, however, on the Tees is but about 600,000 tons per annum, forming only a small proportion of the whole yield of the district. In some cases where the iron works are conveniently situated, the slag is carried out by barges and tipped on to banks at high water to form training walls or for reclaiming land, being thus got rid of. But, as a rule, the labor and capital expended upon this unproductive substance tell heavily upon profit. No wonder, then, that from the first persistent efforts have been made either to utilize it or to get rid of it altogether. In early times slag was broken up by hand and used for road-making, and it so continues to be used where it can be had without a heavy cost for transport; but there is only a limited demand for it for this purpose. On the Continent, where stone is scarce, slag plays a prominent part in road-making, as in Silesia and other similarly situated districts. Another direction in which many attempts have been made to utilize slag, both at home and abroad, is to adapt it

for constructive purposes, and various schemes have been devised for transforming the highly refractory slag into bricks, sand, and other materials for building. Some of these schemes have proved successful within certain limits; but the peculiar nature of the slag has more generally led to failure, owing either to the difficulty of dealing with it, or the attendant expenses. Among the most prominent living scientific investigators of the question was Mr. Bessemer, and about fifteen years since Mr. John Gjers devised a method of granulating slag, the sand produce being used in place of silicious sand on the pig beds. The practice, however, was discontinued after a time for technical reasons. Several other practical men have taken an active part in endeavoring to solve the slag difficulty, among them being Mr. D. Joy, Mr. T. Bell, Mr. Lürmann, and Mr. Homer. Some time since Mr. Charles Wood, of Middlesbrough-on-Tees, directed his attention to the utilization of this unproductive material, and after about five years of careful study, experiment and practical research, he has succeeded in effecting the conversion of blast-furnace slag into various forms, and in applying it to several industrial purposes upon a practical and commercial scale. At the time when he started upon his investigations, there was no instance of slag being manufactured into a commercial commodity in this country, its only known application being that of road-making. Mr. Wood, however, has succeeded in utilizing it for the manufacture of building bricks, concrete, cement, mortar, and slag wool. The various processes of conversion and manufacture are carried on under Mr. Wood's management at the Cleveland Slag Works at Middlesbrough, which, together with the adjoining Tees Iron Works, belong to Messrs. Gilkes, Wilson, Pease & Co., of which latter works Mr. Wood is also the manager, and whence the slag is obtained. In following the highly interesting processes of conversion consecutively, we must first take our readers to the iron works, where the slag is run from the blast furnaces into two different machines, one of which produces a coarse kind of shingle, and the other a fine sand. For making shingle the liquid slag is run direct from the blast furnaces on to a circular, horizontal, rotative table composed of thick slabs of iron kept cool by having water circulated through them. The table, which revolves slowly, carries the slag around to a certain point, by which time it has solidified. At that point it encounters a stream of water, which further cools it, and soon after it comes against a set of scrapers, which break it up and clear it off the table, delivering it into wagons placed below, and which convey it away. For producing slag-sand, the slag is run from the blast furnace into a hollow wheel revolving upon a horizontal axis, and fitted with iron buckets inside. A bath of water is maintained inside the wheel at the bottom, and is kept in a state of violent agitation by the revolving action. As the molten slag enters the body of water it is immediately disintegrated and assumes the form of sand, the water taking up the heat from the molten slag and giving it off in the shape of steam. A constant flow of water is maintained into the machine, and the sand is separated from it and elevated to the top of the machine by the bucket plates, which are perforated. Arrived at the upper part of the machine, the slag-sand is dropped into a spout, and thence finds its way into wooden wagons, by which it is conveyed to the slag works for manufacture.

The slag works occupy a main building 120 feet long, 50 feet wide and five stories high, with basement beneath, and engine-house, boiler-house, and other accessories annexed. This building was constructed of slag cement concrete, composed of four parts of slag-shingle to one part of cement, and it forms a very solid and comparatively indestructible structure. The slag-sand is brought here from the blast furnace and is tipped into stores below, whence it is elevated to the top floor by means of a hoist, which is fitted with an ingenious automatic safety brake designed by Mr. Wood. The special manufacture in this building is that of bricks, and in carrying this out two machines are used, one having been designed by Mr. J. J. Bodmer and the other by Mr. Wood. For the Wood machine the sand is delivered into a hopper through a coarse screen, which retains any pieces of slag

or other substances which may have found their way into the sand. Arrived at the floor below, the sand is automatically measured on a revolving cylinder, divided on the outside, and placed at the bottom of the hopper. From another hopper selenitic lime in powder is also measured by a similar contrivance, and the two substances unite in one shoot, where they become mixed in the proportion of ten parts of sand to one of lime. The mixture is carried down a hopper into the pug-mill of the brick-making machine, where the two substances are further incorporated. This machine was designed by Mr. Wood, and is the outcome of considerable experience with another machine to which we shall presently refer, and which it has to some extent superseded. Mr. Wood's brick machine has a horizontal, circular, rotating moulding table, which contains six pairs of moulds, four bricks being pressed at the same time. During the time of pressing—which is effected by direct mechanical pressure—the table remains stationary, and the same time four other moulds are being filled and the remaining four are delivering the pressed bricks. As they are delivered they are taken off the machine by two girls and are removed to an air-drying shed—the machine producing from 11,000 to 12,000 bricks per day. There they remain for a week or ten days, after which they are stacked in the open air to harden, which occupies another five weeks or so, when the bricks are ready for the market. The bricks thus produced are very tough; they do not split when a nail is driven into them, and are largely used for interior work, for which they are well adapted from the regularity of their surface and other qualities. They find a good market in London, and are not subject to breakage in transit. According to a certificate recently issued from Kirkaldy's testing works, some bricks taken from a stock three years old were not crushed until a pressure of 21 tons had been reached. Others taken from a stock four months old were crushed with 9 tons pressure, thus showing not only unusual toughness and strength, but that they were greatly improved by age. We thus have the curious anomaly of bricks being made without burning and of a wet season being favorable to the hardening process. The second machine at the Cleveland Slag Works is that of Mr. J. J. Bodmer, and was the first one put up at the works. It is worked by hydraulic power, and has a horizontal revolving table with twelve moulds. The slag-sand and the lime are mixed on their way to the machine, but the machinery for effecting the mixing is more complex than that used for Mr. Wood's machine. The rate of production in the Bodmer press is about the same as in the Wood machine, the distinctive difference between the two presses being that the former is worked by hydraulic power, and the latter by direct mechanical pressure, Mr. Wood's machine possessing several advantages over that of Mr. Bodmer. In another department the manufacture of artificial stone is carried on, the stone being moulded into chimney-pieces, window-heads, balustrading, and outside ornamental builders' work generally. The stone is composed of two-and-a-half parts of finely pulverized slag and two-and-a-half parts of ground brick to one part of Portland cement. The mixture is run into moulds and sets quickly, the articles being ready for the market in four or five days. Besides bricks and stone articles, the slag is used for making mortar, cement, and concrete. The mortar is a mixture of slag and common lime, the cement being composed of the same materials with the addition of iron oxides. Slag cement also forms the subject of a recent invention by Mr. Frederick Ransome, who has produced some very remarkable results. His cement consists of a mixture of slag-sand and carbonate of lime in the proportion of two parts of lime to one part slag-sand. These are burnt together, and experiments show the result to be a cement possessing nearly 30 per cent. greater strength than Portland cement.

Perhaps the most beautiful, and certainly not the least remarkable, outcome of blast furnace slag is slag-wool, or silicate cotton as it is also called, owing to its resemblance to cotton-wool. The process originated, we believe, with Messrs. Siemens Brothers, on the Continent, and the manufacture has been before attempted in England, but, as far as we are aware, has not succeeded. As carried out by Mr. Wood, at the Tees Ironworks, a

jet of steam is made to strike against the stream of viscous molten slag as it is run off from the blast furnace. This jet scatters the molten slag into a stream of shot which is projected forwards near the mouth of a large tube, in which a couple of steam jets cause an induced current of air. This tube opens into a receiving chamber, composed chiefly of wire gauze, and measuring about 33 feet long, by 15 feet wide, and 12 feet high. As each shot leaves the stream of slag it carries a fine thread or tail with it. The shot, being heavy, falls to the ground, while the fine wooly fibre is sucked through the tube and deposited in the chamber. The appearance of this chamber after a charge has been blown into it is singularly beautiful. Not an inch of floor, sides, or roof but is covered with a thick layer of the downy silicate cotton, bringing forcibly to mind the familiar words of the old 147th Psalm—

"Large flakes of snow like fleecy wool."

After each blowing the wool is removed by forks, and packed in bags for consignment to a London firm—Messrs. Daniel Dade & Co.,—who make it into mattresses which are used for covering steam boilers, and for other purposes where it is desired to prevent the radiation of heat. For this purpose slag-wool is eminently adapted, as it is a very bad conductor of heat, and is, moreover, perfectly incombustible. The make of slag-wool at the Tees Works is about three tons per week, and as during the running of a 4-ton slag ball about 1¼ cwt. of slag-wool is made, it follows that for producing these three tons nearly 200 tons of slag have to be operated upon. Another useful purpose for which blast-furnace slag has been successfully utilized is that of glass manufacture. The vitreous character of slag indicates a resemblance to glass in its composition. It does, in fact, contain the principal components of glass, but not in proper proportions, and those in which it is deficient have therefore to be added with others which are not present. Some years since Mr. Bashley Britten investigated this question, and in the end succeeded in utilizing for the manufacture of glass, not only the material, but the heat of the slag. This latter is a very important point, inasmuch as upon it depends the economy of the utilization, and, therefore, its commercial success. The practical result of Mr. Britten's researches was the establishment by a company of some works at Finedon, in Northamptonshire, where the manufacture of glass bottles from slag is now, and has for some time past, been regularly carried on. The glass works are situated in close contiguity to the blast furnaces of the Finedon Ironworks, where the Northamptonshire ore is worked, and as the molten slag is run from the furnaces it is conveyed in carriers to the glassworks. In these works a Siemens regenerative gas furnace applied to a glass-melting tank enables the preparation of the "metal" to be carried on continuously, affording a constant supply to the glass blowers. The ingredients of the glass are fed into the tank in charges of about 500 lbs., the larger half of which is the molten slag, the remainder being the other necessary ingredients, such as sand and alkalis. In the tank these substances are fused and fined, the fused metal flowing through a bridge to the other end of the tank, where there are five working holes, from which the metal is taken by the workmen and fashioned into useful articles in the usual way. For the present the manufacture is confined to wine and beer bottles, of which about ninety gross can be produced per day. So far the results have proved sufficiently satisfactory to induce the company to extend their works, plans for which are in course of preparation. It is proposed to erect not only additional furnaces, but plant for the manufacture of other articles besides bottles, and for these a wide field opens itself. The glass produced is said to be stronger than ordinary glass, and the color can be varied as required, the natural tint being green. Its working qualities are said to be of the highest order, as it comes from the furnace in the best possible condition for the worker. Some bottles made at Finedon were sent to the Paris Exhibition, last year, where they obtainedho norable mention, a testimony at once to their character. A new method of toughening glass has recently been discovered by Mr. Frederick Siemens, of Dresden, and it is proposed to apply this process to slag glass for the purpose of manufacturing

railway sleepers and other articles. Details of Mr. Siemens' process are not at present to hand, but, judged by results, it would appear to differ from that of M. de la Basties, inasmuch as when the toughened glass is broken it does not fly into minute atoms as does de la Bastie's, but simply fractures, somewhat similarly to cast iron.

We have now taken our readers through the various interesting and ingenious processes which are being carried on as ordinary commercial pursuits. The successful utilization of slag has a double importance—it not only helps to produce the annual accumulation of a cumbersome and worthless waste product, but it adds new branches of manufacture to the industrial arts. Mr. Wood may be complimented upon his perseverance and congratulated upon his success. When in full work, 450 tons of slag are produced per day at the Tees Works, and of this quantity about 1,000 tons per month are converted into sand for brick making, the average make with the two machines going being 110,000 bricks per week, the whole of which now find a ready sale in the London market. We should mention that slag bricks are also being make at the Moss Bay Ironworks, by Messrs. Kirk Brothers, who reduce the slag to powder first under edge runners, and then pass it between millstones. The powder is then moistened and pressed into bricks, which are hardened in the open air. The bricks are very good, but they are heavy, and are said to be expensive. At the Acklam Ironworks, blocks for paving streets, stables, and the like, are being made from slag. The slag is there run into heated moulds, and after each block is formed, it is removed from the mould and placed in an oven to anneal. These blocks are heavy, but wear well. In view of the general usefulness of slag when converted into the various articles we have described, it is to be hoped, in the interests of commerce and progress, that the practice of its utilization may become more and more extended. Doubtless human progress will show that what is now the veriest waste will, in the course of time, assume a condition of value. Thus will art be made to approximate to nature in that it will know no waste.—*The Times.*

A NEW RATIONAL FORMULA FOR PILLARS.

BY JOHN D. CREHORE.

Contributed to VAN NOSTRAND'S ENGINEERING MAGAZINE.

Having given in this Magazine for October, 1878, a method of finding empirical formulæ for the strength of pillars, I now offer, as the result of further investigation, a formula resting upon theoretical grounds alone, verified by a limited number of experiments, and, so far as I know, new, not in its form, but in the mode of its derivation, in the number and character of its variables, and (for pillars fulfilling its conditions) in its freedom from empirical constants requiring for their determination the destruction of many pillars.

In other words, an attempt is here made to establish a formula for the strength of pillars that yield by bending rather than by direct crushing, in terms of the *modulus of transverse elasticity* of the material, and of its *ultimate resistance to compression.* If this effort prove successful we shall have a formula at once applicable to all materials used for such pillars, struts, or columns, of whatever cross-section, and testing will be reduced to the simple determination of these two factors from small specimens, without the expensive destruction caused by testing the finished pillar.

And, although for certain materials, as wrought iron, for instance, the greatest resistance to direct crushing, has not been very definitely ascertained, yet as Mr. William Kent has pointed out, this indefiniteness has resulted less from inability to make the correct determination, than from the want of uniformity of method on the part of those who have treated the subject. With a proper definition of the term "ultimate resistance to compression," there can be no great difficulty in its determination from standard specimens by means of a suitable machine.

The "modulus of elasticity" also—

although it is in one sense an imaginary number, truly expresses the relation of magnitudes which are definite and measurable.

Only pillars of uniform cross-section are here considered.

Let l= length of pillar.
h= least diameter.
r= least radius of gyration of cross-section.
S= area of cross-section.
D= greatest deflection of pillar, all in inches.
E= modulus of transverse elasticity in lbs. per square inch.
$I = S r^2$= least moment of inertia of cross-section.
C= crushing strength of standard specimen in lbs. per square inch.
P= breaking weight applied at the end of the pillar and in the line of its axis before deflection, in lbs. per square inch.
$Q = \frac{P}{S}$= breaking weight per square inch of cross-section.
M_x= moment of forces devoloped in any normal cross-section by the given load, P.
M_1= the end moment at the lower end when that end is fixed.
M_2= the end moment at the top when the upper end is fixed.

Suppose the pillar vertical, and take the origin of rectangular co-ordinates at the lowest point of the pillar's axis, which call also the axis of x. Then the ordinary expression for the moment at any cross-section, is,

$$M_x = -EI\frac{d^2y}{dx^2} = \frac{M_1 - M_2}{l}x - M_1 + Py, \quad (1)$$

wherein no account is taken of the modified condition of every cross-section, due to the longitudinal pressure Q per unit.

Now, since the full unit-strength of the cross-section of the unloaded pillar is C, and the remaining unit-strength of the loaded pillar's cross-section is (C−Q), it follows that the expression for the moment of the internal forces, developed in any cross-section, must be diminished in the ratio $\frac{C-Q}{C}$. We then have

$$M_x = -P\varepsilon^2\frac{d^2y}{dx^2} = \frac{M_1 - M_2}{l}x - M_1 + Py, \quad (2)$$

if $$\varepsilon^2 = \frac{EI(C-Q)}{PC} = \frac{Er^2(C-Q)}{QC}.$$

There will be three cases, according as we consider neither, both, or one only, of the ends fixed.

I. If neither end can produce any moment, $M_1 = M_2 = o$, and we have from equation (2)

$$\varepsilon^2\frac{d^2y}{dx^2} = -y \quad . \quad . \quad . \quad (3)$$

Multiplying by $2dy$,

$$2\varepsilon^2\frac{dyd(dy)}{dx^2} = -2ydy.$$

Integrating this equation and putting a^2 for the arbitrary constant of integration,

$$\varepsilon^2\frac{dy^2}{dx^2} = a^2 - y^2.$$

From which

$$\frac{dx}{\varepsilon} = \frac{dy}{(a^2-y^2)^{\frac{1}{2}}}.$$

Integrating again between the limits, for x, o and l, and for y, o and o,

$$\therefore\ l = \varepsilon\left\{\sin^{-1}\frac{y}{a}\right\}_0^0 = \varepsilon n\pi;$$

where n may be any whole number; but in order that P may have the least value it can have, consistent with the bending of the pillar, necessarily assumed in establishing equations (1) and (2), n must be equal to unity. (See *Rankine's Applied Mechanics*, page 352). Therefore

$$l^2 = \pi^2\varepsilon^2 = \frac{\pi^2 Er^2(C-Q)}{QC},$$

$$Q = \frac{C}{1 + \frac{Cl^2}{\pi^2 Er^2}}; \quad . \quad . \quad . \quad (A)$$

which is the formula for pillars with rounded ends that can generate no end-moments.

II. If both ends of the pillar are equally fixed, so that the elastic curve at each end after flexure, has for its tangent the original undeflected axis, then in equation (2) $M_1 = M_2$, whence

$$P\varepsilon^2\frac{d2y}{dx^2} = M_1 - Py \quad . \quad . \quad . \quad (4)$$

Multiplying by $2dy$, equation (4) becomes

$$2P\varepsilon^2\frac{dyd(dy)}{dx^2}=2M_1dy-2Pydy.$$

Integrating we find

$$P\varepsilon^2\frac{dy2}{dx^2}=2M_1y-Py^2+a \quad . \quad . \quad . \quad (5)$$

where a, the arbitrary constant, must vanish, since $\frac{dy}{dx}=o$ when $y=o$.

Hence from (5)

$$\frac{dx}{\varepsilon\sqrt{P}}=\frac{dy}{(2M_1y-Py^2)^{\frac{1}{2}}}.$$

Integrating again with the condition that $y=o$ when $x=o$, there results, after canceling $\sqrt{P}$ from the denominators,

$$\frac{x}{\varepsilon}=\sin^{-1}-\left\{\frac{2Py-2M_1}{2M_1}\right\}+\frac{\pi}{2} \quad . \quad . \quad . \quad (6)$$

Also we have $y=o$ when $x=l$, so that (6) becomes,

$$\frac{l}{\varepsilon}=\sin^{-1}\left\{-1\right\}+\frac{\pi}{2},$$

$$=\frac{3\pi}{2}+\frac{\pi}{2}=2\pi,$$

to be consistent with the permanence of l, and with the least positive value of P. Therefore

$$l^2=4\pi^2\varepsilon^2=\frac{4\pi^2Er^3(C-Q)}{QC},$$

$$Q=\frac{C}{1+\frac{Cl^2}{4\pi^2Er^3}}, \quad . \quad . \quad . \quad . \quad (B)$$

which is the formula for pillars with both ends equally and fully fixed.

III. When one end of the pillar is fixed, and the other end can cause no end-moment, we have (say) $M_1=o$, and derive from equation (2),

$$P\varepsilon^2\frac{d^2y}{dx}=\left\{\frac{M_2}{l}x-Py\right\}dx; \quad . \quad . \quad . \quad (7)$$

an equation whose second member cannot be integrated without specific connection between x and y, unless there is such a relation among the quantities which compose it, that the member shall be wholly a function of x. (*See DeMorgan's Dif. and Int. Calculus*, page 208.)

Not knowing the specific connection between x and y, nor what relation among its component quantities there may be, to render the second member a "function of x only," I shall here assume a connection expressed by the equation of a parabola, viz.,

$$y=x+bx^2, \quad . \quad . \quad . \quad . \quad (8)$$

where, since $y=o$ when $x=l$,

$$b=-\frac{1}{l};$$

and shall attempt only an approximate solution for this third case.

Putting this value of y into equation (7), it becomes

$$P\varepsilon^2\frac{d2y}{dx^2}=mx-Pbx^2, \text{ if } m=\frac{M_2-Pl}{l}$$

Integrating with the condition that $\frac{dy}{dx}=o$ when $x=l$, since the upper end of the pillar is now fixed,

$$\therefore P\varepsilon^2\frac{dy}{dx}=\tfrac{1}{2}m(x^2-l^2)-\tfrac{1}{3}Pb(x^3-l^3) \quad . \quad . \quad (9)$$

Integrating again between the limits o and y, o and x,

$$\therefore P\varepsilon^2y=\tfrac{1}{2}m\left(\frac{x^3}{3}-l^2x\right)-\tfrac{1}{3}Pb\left(\frac{x^4}{4}-l^3x\right) \quad (10)$$

But $y=o$ when $x=l$, hence from (10)

$$m=\tfrac{5}{4}Pbl. \quad . \quad . \quad . \quad (10)$$

If in (9) we make $\frac{dy}{dx}=o$, we may find the value of x which renders $y=D$ a maximum. Dividing by $(x-l)$, and by means of (11), eliminating m and Pb, we derive from (9) when y is a maximum,

$$x=\tfrac{1}{16}l(1\pm\sqrt{33}),$$
$$=0.42153l$$

since the negative value is not here admissible.

Taking $x=0.42153l$, $y=x+bx^2$, and $b=-\frac{1}{l}$, we find from (10) after restoring ε^2 and m, and reducing,

$$Q=\frac{C}{1+\frac{Cl^2}{22.511Er^2}}=\frac{C}{1+\frac{Cl^2}{2.28\pi^2Er^2}}, \quad . \quad . \quad C$$

which is an approximate formula for pillars having but one end fixed.

The reader must have observed the identity in the *form* of equations (A), (B), and (C), with that of the well-known Gordon Formula, especially as modified by Rankine's introduction of r^2 as a function of h^2, and with Hodgkinson's supplementary formula.

There is also a gratifying identity in the relation of the constants peculiar to

the condition of the pillar, as herein found analytically, and the relation of the corresponding constants as deduced from experiments of Hodgkinson and others.

These constants as determined above are proportional to the numbers here placed below the letters marking their respective formulæ;

(A) . (B) . (C)
1 . 4 . 2.28

in the *denominator* of the denominator. And in Rankine's *Rules* and *Tables*, page 210, for the corresponding cases we find

$$4 \cdot 1 \cdot \tfrac{16}{9},$$

in the *numerator* of the denominator. Inverting these last that they may become factors in their respective denominators, we have

$$\tfrac{1}{4} \cdot 1 \cdot \tfrac{9}{16}.$$

Multiplying these by 4, for comparison with the first set, we get the proportionals 1 . 4 . 2.25.

This comparison of constants affords a presumption that the approximation in formula (C) is practically near enough to the exact solution. In fact, the importance of even this degree of exactness, in the relation of these three constants, seems not great, when we reflect that, in practice, we seldom know to what degree a given strut really fulfills one of these three conditions; that is to say, we cannot often tell the exact amount of influence which the end-bearings, or fastenings, exert upon the stability of the strut or pillar. For example, the experiments in Table I below, numbered 16, 17, 13, 14, 11, 5, 21, for pillars having hinged or round ends, or pin-bearings, are better represented by the formulæ for pillars with one end fixed, than by the formulæ for those incapable of developing end-moments.

1. *Wrought Iron.*—Let us now apply our formulæ to actual tests of the strength of columns, so far as we can in the present state of uncertainty in regard to the values of the variables C and E; and first take wrought iron columns as tested for the Cincinnati Southern Railway, and reported by Mr. Thomas D. Lovett, who says: "More of these tests were made than would have been necessary if the formula (Gordon's) adopted in the specifications, for the determination of the sectional areas of columns, had been of universal authority; but though the best yet proposed, doubts were expressed by competent bridge-builders and engineers, as to its correctness when applied to columns of certain shapes, and it was thought best to make a number of experiments sufficient to test the ormula itself."

In that report we find Gordon's formula,

$$\frac{P}{S} = \frac{f}{1 + \frac{l^2}{3000h^2}},$$

both ends flat.

$$= \frac{f}{1 + \frac{l^2}{1500h^2}},$$

one flat, one round end.

Rankine's formula,

$$\frac{P}{S} = \frac{f}{1 + \frac{l^2}{36000r^2}},$$

both ends flat.

$$= \frac{f}{1 + \frac{l^2}{18000r^2}},$$

one flat, one round end.

The Gordon and Rankine formulæ are identical for columns of rectangular cross-section, and in all cases where $h^2 = 12r^2$.

From the values of P found by experiment, f in these formulæ was computed and its values tabulated in the report. And since no test is therein given for destruction by direct crushing, without buckling or bending, we shall be obliged in these cases to determine C in formulæ (B) and (C), in the same manner by computation.

Table I shows the results of this computation of the values of C, arranged beside the reported values of f; in both cases pillars with hinged ends being treated as if having one end fixed.

In Table I the mean value of E has been used for Nos. 2 and 4. Also, the least value of r^2 has been employed in Nos. 13 and 19, although the report states that owing to "defective fittings at the ends, inadequate balancing of the columns in the center, or insufficient

riveting," these two columns yielded anomalously in the other direction. But since they must have deflected in the direction of least resistance, it is plain that no value of r^2 greater than the least, should be used in computing Q or C.

The mean values of f and C, in Table I, agree remarkably well with Rankine's mean value of the "resistance of wrought iron to crushing by direct thrust," viz., "about 36,000 to 40,000 lbs. per square inch;" notwithstanding the language of the Cincinnati Southern Report, to the effect that f is "very approximately proportional, but not equal, to the ultimate strength of the iron."

In regard to the nature of this f in the numerator of the Gordon formula, Rankine is very explicit when he says: "One part of the intensity of the greatest stress on the material, is simply the intensity,

$$p' = \frac{P}{S},$$

due to the uniform distribution of the load over the section. Another part, p'', of the greatest stress, is that which arises from the lateral bending;" and, "The whole intensity of greatest stress on the material of the pillar, being made equal to a co-efficient of strength, f is expressed by the following equation:

$$f = p' + p''."$$

Now, since f cannot be less than p' which is positive, p'' must be equal to zero, or else be positive also; so that p' and p'' represent the same kind of stress. But p' by definition denotes compressive stress; therefore p'' must stand for resistance to compression also, and of course the sum f is of the same kind as its parts.

Theoretically, then, f in the Gordon and Rankine formulæ, stands for greatest stress in compression. But since in each of those formulæ, there are *two* constants to be determined by experiment upon finished pillars, it follows that any value may be assigned to one of these constants, consistent with the derivation of the other, as a constant, from experiment. And hence in these two formulæ, f may be replaced by numbers not truly representing the crushing strength of the material. The truth of this statement is illustrated by equations (12) and (13) used in computing the last two columns in Table II, from data found in Stoney's *Theory of Strains*, and there arranged from Hodgkinson's experiments on solid rectangular struts of wrought iron. In applying the Gordon formula to the set of tests in Table II, I have employed the constants assigned by Mr. Stoney for the same, viz.:

$$Q = \frac{P}{S} = \frac{35840}{1 + \frac{l^2}{3000 h^2}}, \quad \ldots \quad (12)$$

But unfortunately for the application of formula (B), neither C nor E is given for these Hodgkinson experiments, and we are left to infer a value of C from the behavior of the two shortest bars in the list. The shortest bar, for which $l \div h = 3.65$, "bore 22.5 tons = 50,400 lbs. per square inch without fracture." No. 33, where $l \div h = 7.331$, was broken by 48,682 lbs per sq. inch. I have therefore assumed C = 50,000, and taken E, the modulus of transverse elasticity, at its mean value given in Table I, and do not assert (further than an inference is to be drawn from the agreement of computed and experimental values of Q) that these were the actual values of C and E for those irons in table II. Formula (B) then becomes in this case,

$$Q = \frac{50000}{1 + \frac{l^2}{1797 h^2}}, \quad \ldots \quad (13)$$

since $h^2 = 12\, r^2$, for the rectangular section, and both ends are fixed.

Table II shows that for 15 out of the 21 tests, formula (B) gives the value of Q nearer to its experimental value than does the Gordon formula with Mr. Stoney's constants.

It is evident that the Gordon formula is not applicable to those pillars whose ratio of length to least diameter is such that Q is greater than the value assigned to f; while there is no such failure in the application of equations (A), (B) and (C), since in these Q cannot be greater than C, the known ability of the standard specimen of the material, to resist crushing.

It would seem, therefore, that these new formulæ (A), (B) and (C), involving the variables C and E together with $(l^2 \div h^2)$ or $(l^2 \div r^2)$, must be more flexible

and versatile than the Gordon formula involving only the variable $(l^2 \div h^2)$, or Rankine's modification involving only the variable $(l^2 \div r^2)$.

2. *Cast Iron.*—For illustrating the application of equations (A) and (B), to pillars of cast iron, let us take Col. Merrill's tables as given in his well-known work on *Iron Truss Bridges for Railroads*, dividing the gross experimental breaking weights therein given, by the area of the pillar's cross-section for the unit pressure Q.

His tables are derived from Hodgkinson's experiments upon solid cylindrical cast iron pillars, having both ends flat or both round, and the formulæ used, are, according to our notation—

a. Hodgkinson's, for length not less than 30 diameters.

Flat ends,

$$P = 98922 \frac{h^{3.55}}{(\frac{1}{12} l)^{1.7}} \quad . \quad . \quad . \quad (14)$$

For lengths not less than 15 diameters.

Rounded ends

$$P = 33379 \frac{h^{3.76}}{(\frac{1}{12} l)^{1.7}} \quad . \quad . \quad . \quad . \quad (15)$$

And for lengths less than 30 and 15 diameters respectively, Hodgkinson corrects the value of P found from (14) or (15), the corrected value being

$$P_1 = \frac{PCS}{P + \frac{3}{4}CS} = \frac{CS}{1 + \frac{3CS}{4P}} \quad . \quad . \quad (16)$$

C being called 109,801 for cast iron.

b. Gordon's Flat ends,

$$Q = \frac{P}{S} = \frac{80000}{1 + \frac{l^2}{400h^2}} \quad . \quad . \quad . \quad . \quad (17)$$

Rounded ends

$$Q = \frac{80000}{1 + \frac{l^2}{100h^2}} \quad . \quad . \quad . \quad . \quad (18)$$

The last column of table IV is computed from the equation

$$Q = \frac{80000}{1 + \frac{l^2}{125h^2}} \quad . \quad . \quad . \quad . \quad (19)$$

a correction manifestly needed, since all the differences as computed by equation (18) are negative.

For E=12,215,000, see Stoney's *Theory of Strains*, page 180.

Eleven of the 22 tests in Table III, are represented better by (B) than by the Gordon formula, even with the fixed values of C and E, here employed, instead of values to be derived in practice, from standard specimens.

Of the 25 experiments in Table IV, formula (A) represents 18 better than the Gordon formula, 14 better than equation (19), and only 10 better than Hodgkinson's, with his supplementary equation for values of $l \div h$ not greater than 15.

It should be kept in mind that Hodgkinson deduced his formulæ from his own experiments, and therefore there ought to be a close agreement between the two.

For Table IV, as for Table I, formula (A) has been modified in order to take account of the end moments unavoidable with a metal so compressible as cast iron, and as used in Table IV, it is,

$$Q = \frac{109801}{1 + \frac{109801 l^2}{1.25\pi^2 E r^2}} = \frac{109801}{1 + \frac{l^2}{85.7782 h^2}} \quad . \quad (A_1)$$

since here E=12,215,000, and $h^2 = 16r^2$.

It will be perceived that, in general, the difference between the experimental value of Q for cast iron, and the value as computed by any one of the formulæ, is not greater than the differences which appear in the experimental values themselves, for pillars having the same *ratio* of length to diameter, but of different diameters. From Nos. 58, 59; 61, 63. 66, 67; 68, 69; 72, 73; 79, 80; 81, 82, 83; 86, 87; 97, 98, we might almost infer the law which has been pointed out by various writers, viz., that the outer crust of cast iron as it comes from the foundry, is stronger than the mass within, and hence small columns of that metal are relatively stronger than those of larger diameter. But the number of experiments is not sufficient to enable us to modify the formulæ so as to represent this property of cast iron.

3. *Steel.*—If for "mild" steel we call E=30,000,000, and C=200,000, formula (B) becomes,

For cylindrical posts,

$$Q=\frac{200000}{1+\frac{l^2}{370.11h^2}}$$

For rectangular posts,

$$Q=\frac{200000}{1+\frac{l^2}{493.48h^2}}.$$

And for "strong" or hard steel, calling E=36,000,000, and C=300,000, (B) gives
For cylindrical posts,

$$Q=\frac{300,000}{1+\frac{l^2}{296.09h^2}}$$

For rectangular posts,

$$Q=\frac{300,000}{1+\frac{l^2}{394.787h^2}}$$

Now Stoney has given on page 279, Mr. B. Baker's "values for the coefficients in Gordon's formula when applied to solid steel pillars," with flat ends. With those coefficients Gordon's formula for solid round pillars of mild steel is,

$$Q=\frac{67200}{1+\frac{l^2}{1400h^2}} \quad . \quad . \quad . \quad (20)$$

For strong steel,

$$Q=\frac{114240}{1+\frac{l^2}{900h^2}} \quad . \quad . \quad . \quad (21)$$

For solid rectangular pillars with flat ends, Mr. Baker's equations are—
For mild steel,

$$Q=\frac{67200}{1+\frac{l^2}{2480h^2}},$$

For strong steel,

$$Q=\frac{114240}{1+\frac{l^2}{1600h^2}};$$

in which the coefficients in the denominators, bear to the corresponding coefficients he has given for cylindrical pillars, the relation of 16 to 9, instead of 16 to 12, the ratio of the squares of their radii of gyration for the cross-sections. This is manifestly incorrect, unless there is a difference in the character of the steel, due to the form into which it is rolled or moulded, as indeed is, doubtless, the case. But as this consideration would change the values of C and E in my own equations, I have not hesitated, for the purpose of this comparison, to replace the coefficients 2480 and 1600, by $\frac{4}{3}\times1400=1867$, and $\frac{4}{3}\times900=1200$, respectively, as required to adapt equations (20) and (21) to solid rectangular pillars. And with only this change, the foregoing equations for steel, have been used in computing Table V.

From Table V it is evident that Mr. Baker's coefficients are not applicable for values of $(l\div h)<50$; for it cannot be that steel columns only 10 diameters in length would be crushed or destroyed by 62,720 lbs. per inch, even if the ultimate strength of standard specimens were not greater than 100,000 lbs; hence, the series of values of Q, must increase faster from 50 diameters down to 10, than his does, or else somewhere exhibit inadmissible breaks before reaching its proper limit.

Hodgkinson felt this difficulty with his own formulæ, and met it by means of the supplementary equation (16), where he has put CS into the numerator and denominator, and anticipated the form of the Gordon equation, and that of the three equations (A), (B), and (C), determined above from theoretical considerations alone.

4. *Wood.*—For rectangular pillars of "timber, having flat capitals and bases," Rankine makes Gordon's formula,

$$Q=\frac{P}{S}=\frac{7200}{1+\frac{l^2}{250h^2}}.$$

Rankine also gives "on the authority of Mr. Hodgkinson's experiments, for the *ultimate* resistance of posts of *red pine* to crushing by bending,"

$$Q=500C\,\frac{h^3}{l^2}$$

according to our notation; a formula to be used only when it gives Q a value less than C.

Mr. Stoney tabulates Rondelet's proportionals for "the force required to break a timber pillar with fixed ends," and Mr. R. P. Brereton's "breaking-weights," for fir or pine, determined for "large pillars of soft foreign timber, with their ends adjusted in the ordinary man-

ner." These will be found in Table VI below.

According to the "*Pocket Companion*" of Carnegie Brothers & Co., "Capt. C. Shaler Smith's formula for the breaking weight of either square or rectangular pillar of moderately seasoned white and yellow pine, with flat and firmly fixed ends, equally loaded," is

$$Q = \frac{5000}{1 + \frac{l^2}{250h^2}}$$

where 5,000 is called the breaking load in pounds per square inch, "in short blocks."

If for pine we give C the low value for 5,000, and take E=1,460,000, Rankine's lowest value, we shall find by means of our formulæ (A), (B) and (C), the values of Q, as shown in the last three columns of Table VI, for pillars that absolutely fulfill the conditions imposed upon our oiginal equation (2).

But the reader will easily see that our inability to say how nearly those conditions are realized in a given case, leaves us still in need of some experiments on finished pillars, to determine the effect of differently conditioned ends. In the meantime it will be wise practice (as seen in the third set of values of Q in Table VI.), to place small dependence upon assistance to come from the ends, especially of such materials as moderately seasoned pine; unless the deflected axis of the pillar is known to have its tangent at the end truly in the line of the original undeflected axis.

TABLE I.—WROUGHT IRON PILLARS—LOVETT'S REPORT.

"Phœnix," Hollow Cylinder, four flanged segments riveted.
"American," two flanged bars riveted to the flanges of an I beam.
"Keystone," Octagonal tube, four flanged segments riveted.
"Square," two plates and two channels riveted.

No.	Name.	Kind of Ends.	h	l	r^2	E 1000000	Q by Experiment.	f by Gordon Formula	f by Rankine Formula	C by Formula (B), (C).
			ins.	ins.	sq. in.					
29	Phœnix...	Flat	8.250	336	8.935	285	36,600	57,500	49,400	62,141
28	" ...	"	8.250	336	8.935	257	34,800	54,600	47,000	61,417
10	" ...	"	8.125	324	8.935	291	31,000	47,400	41,100	45,390
6	" ...	"	8.050	180	8.536	274	37,500	43,700	41,500	43,182
15	American.	"	8.	360	5.388	260	23,700	39,700	39,500	53,304
19	" .	"	9.5	324	8.635	329	27,800	37,500	37,200	37,578
18	" .	"	9.5	240	8.653	236	31,500	37,500	37,300	40,648
27	Keystone .	"	8.625	324	9.798	237	27,800	40,800	36,000	40,783
4	" .	"	9.2	324	10.883		24,100	34,100	30,600	30,725
26	" .	"	9.375	324	11.178	275	27,500	38,400	34,700	36,084
25	" .	"	9.625	324	11.424	281	21,100	29,100	26,500	25,569
30	" .	"	9.5	324	11.464	193	30,000	42,100	37,600	46,914
31	" .	"	9 5	324	11.464	236	25,400	36,100	31,900	33,850
24	" .	"	9.5	324	12.041	265	25,000	34,700	31,100	31,579
9	" .	"	8.85	180	7.833	247	32,000	36,400	35,700	37,026
7	" .	"	8.3	180	9.206	238	30,000	34,700	32,900	33,798
8	" .	"	9.	180	10.353	296	36,900	41,800	40,100	41,895
3	" .	"	9.25	180	10.834	346	28,800	32,400	31,200	30,738
2	" .	"	9.3	60	11.044		33,600	34,100	33,900	33,945
22	Square....	"	7.5	312	9.347	278	30,000	47,300	38,700	41,939
32	"	"	9.25	324	10.909	301	30,200	39,800	38,300	39,977
23	"	"	8.43	288	11.628	289	33,200	46,100	39,800	41,896
16	American.	Hinged .	8.	240	5.479	289	26,700	42,700	42,300	46,976
17	" .	"	10.	240	8.733	231	26,500	36,700	36,200	39,925
13	" .	"	10.75	312	10.092	304	24,000	37,500	31,100	36,276
14	" .	"	10.	312	8.733	260	22,000	36,300	35,600	37,868
11	Phœnix...	Round..	8.125	324	8.935	271	21,700	44,700	35,900	37,289
5	Keystone .	Hinged.	9.22	324	10.954	295	22,000	40,100	33,700	32,239
21	Square....	" .	10.	309	11.000	310	25,560	41,700	37,800	37,350
	Means....					273½		40,190	36,710	39,941

TABLE II.—SOLID RECTANGULAR PILLARS OF WROUGHT IRON.

HODGKINSON'S EXPERIMENTS. FLAT ENDS.

b=breadth, $h^2=12r^2$. E=27,311,111, C=50,000.

No.	b	l	$l+h$	Q by Experiment.	Excess over Q by	
					Gordon's Formula.	Formula (B).
	ins.	ins.				
33	1.023	7.5	7.331	48,682	—13,473	— 134
34	1.023	15	14.663	34,554	— 1,105	+10,103
35	1.023	30	29.326	25,327	+ 2,527	+ 8,489
36	3.000	30	30.121	29,655	— 2,137	+ 3,570
37	3.010	30	39.319	27,767	— 4,115	— 890
38	1.024	60	58.594	17,268	— 555	— 88
39	3.000	90	58.524	19.987	— 3,343	— 2,896
40	2.986	30	59.689	16,853	— 470	— 90
41	5.480	60	60.241	17,698	— 1,479	— 1,138
42	3.010	60	60.303	18,067	— 1,865	— 1,530
43	3.010	60	78.227	12,969	— 1,179	— 1,619
44	3.000	120	79.470	10,165	+ 1,377	+ 914
45	1.024	90	87.891	9,753	+ 273	— 317
46	5.860	90	90.407	9,912	— 289	— 900
47	3.005	90	90.452	9,280	+ 336	— 276
48	2.980	60	118.343	5,653	+ 670	+ 33
49	2.980	60	118.343	5,604	+ 719	+ 82
50	2.995	120	120.603	4,280	+ 1,848	+ 1,218
51	3.010	120	156.658	3,379	+ 525	+ 32
52	2.983	90	179.176	2,410	+ 653	+ 240
53	2.980	120	238.659	816	+ 978	+ 714

TABLE III.—SOLID CYLINDRICAL PILLARS. CAST IRON.

HODGKINSON'S TESTS. ENDS FLAT.

$h^2=16r^2$. E=12,215,000, C=109,801.

No.	$l+h$	Diameter, h	Q by Experiment.	Variation from Q, per cent.		
				Hodgkinson's Formula.	Gordon's Formula.	Formula (B).
		ins.				
54	4	0.52	107,674	— 6	—29	— 3.641
55	8	0.5	88,964	— 3	—21	+ 0.085
56	10	0.777	67,502	+13	— 4	+19.227
57	13	0.768	55,959	+13	— .001	+21.444
58	15	0 5	57,321	+ 2	—11	+ 5.267
59	15	0.785	50,182	+11	+ 1/2000	+20.243
60	15	1	51,248	+ 9	+ 7	+17.741
61	20	0.5	45,485	— 1	—13	— 1.758
62	20	0.775	45,596	— 1	—10	— 1.998
63	20	1.022	38,770	+12	+ 5	+15.257
64	24	0.5	36,644	+ 2	—11	— 3.291
65	26	0.777	32,860	+ 1/4	— 9	— 3.503
66	30	0.51	33,111	—10	—25	—22.497
67	30	1.01	25,350	+ 5	— 3	+ 1.231
68	39	0.77	18,921	— 8	—13	—11.284
69	39	1.56	15,153	+ 6	+11	+10.777
70	40	0.51	18,749	— 2	—13	—14.241
71	47	1.29	12,291	— 3	+ 1/4	— 1.261
72	61	0.5	8,465	+ 6	— 7	—10.881
73	61	0.997	7,990	+ 1/2	— 2	— 5.581
74	79	0.77	5,274	+ 2	— 8	—12.286
75	119	0.51	2,384	+19	— 7	—12.416

TABLE IV.—SOLID CYLINDRICAL PILLARS. CAST IRON.

HODGKINSON'S TESTS. ENDS ROUNDED.

$h^3=16r^3$, $E=12{,}215{,}000$, $C=109{,}801$.

No.	$l \div h$	Diameter, h	Q by Experiment.	Variation from Q, per cent.			
				Hodgkinson's Formula.	Gordon's Formula.	Formula (A,)	Formula (19)
		ins.					
76	8	0.5	76,939	−25	−34	−18.269	−30.937
77	10	0.77	49,280	− 7	−18	+ 2.877	− 3.362
78	13	0.76	38,590	−15	−25	− 4.203	−11.858
79	15	0.497	27,124	+ ½	−11	+11.735	+ 5.336
80	15	0.99	25,660	+10	− 6	+18.110	+11.345
81	20	0.76	20,331	−13	−21	− 4.633	− 6.311
82	20	1.01	19,642	− 9	−19	− 1.288	− 3.029
83	20	1.52	17,928	+ 3	−10	+ 8.149	+ 6.247
84	23	1.29	13,187	+ 5	− 7	+16.175	+15.955
85	26	0.767	14,289	−23	−29	−13.472	−12.632
86	30	0.5	9,697	−13	−19	− 1.464	+ 0.846
87	30	0.99	7,931	+ 9	− 2	+20.477	+23.301
88	31	1.94	7,717	+13	− 3	+16.599	+19.321
89	31	1.96	8,051	+14	− 3	+11.763	+14.371
90	34	1.765	6,360	+ 5	−10	+19.262	+22.737
91	34	1.78	7,058	+ 6	−10	+ 7.467	+10.597
92	39	0.77	5,854	− 5	−17	+ 0.137	+ 3.775
93	39	1.535	5,755	+ ⅓	−16	+ 1.859	+ 5.560
94	40	1.52	5,985	− ⅔	−14	− 6.650	− 3.141
95	47	1.29	4,367	− ½	−18	− 6.000	− 1.901
96	47	1.295	4,149	− ⅓	−18	− 1.060	+ 3.254
97	61	0.5	2,745	− 5	−23	− 9.872	− 5.501
98	61	0.99	2,471	+ 8	−16	+ 0.121	+ 4.978
99	79	0.77	1,675	+ 2	−24	−11.105	− 6.209
100	121	0.5	728	+10	−25	−12.083	− 7.003

TABLE V.—SOLID STEEL PILLARS, WITH FIXED ENDS.

Kind.	$l \div h$	Mild Steel. $E=$ 30,000,000. $C=$ 200,000. Q		Strong Steel. $E=$ 36,000,000. $C=$ 300,000. Q	
		Baker's Formula.	Formula (B).	Baker's Formula.	Formula (B).
Round	10	62,720	157,457	102,816	217,802
"	20	52,267	96,117	79,089	127,608
"	30	40,904	58,280	57,120	74,264
"	40	31,359	37,573	41,127	46,847
"	50	24,123	25,791	30,240	31,768
"	60	18,816	18,645	22,848	22,799
"	70	14,933	14,046	17,727	17,095
"	80	12,061	10,934	14,084	13,266
"	90	9,903	8,739	11,424	10,580
"	100	8,253	7,138	9,433	8,627
Rectangular...	10	63,784	166,301	105,452	239,368
" ...	20	55,343	110,462	85,680	149,017
" ...	30	45,342	70,827	65,280	91,469
" ...	40	36,188	47,144	48,960	59,154
" ...	50	28,730	32,970	37,051	40,751
" ...	60	22,949	24,111	28,560	29,525
" ...	70	18,541	18,303	22,473	24,620
" ...	80	15,176	14,317	18,038	17,355
" ...	90	12,588	11,459	14,741	13,881
" ...	100	10,573	9,406	11,961	11,344

TABLE VI.—SQUARE PINE PILLARS.

$h^2 = 12r^2$. $E = 1{,}460{,}000$, $C = 5{,}000$.

$l \div h$	Breaking Weights, in lbs. per sq. inch.						
			By Formula,				
	Rondelet's Proportionals. Ends fixed.	Brereton's Tests. Ends in ordinary manner.	C. Shaler Smith's. Ends fixed.	Hodgkinson's. Ends fixed.	(*A*) No End-moment.	(*C*) One End fully fixed.	(*B*) Both Ends fully fixed.
1	5,000			5,000			
12	4,167		3,176	5,000	3,126	3,959	4,349
24	2,500		1,513	4,940	1,471	2,437	3,126
36	1,667		809	1,929	782	1,485	2,135
48	833		489	1,085	462	960	1,471
60	417		325	693	313	660	1,076
72	209		230	483	221	478	642
10		1,867	3,571	5,000	3,530	4,228	4,529
20		1,789	1,923	5,000	1,876	2,889	3,530
30		1,400	1,087	2 777	1,053	1,891	2,581
40		1,244	676	1,563	653	1,273	1,875

PHYSICAL CHANGES IN IRON AND STEEL.

From "The English Mechanic."

AMONGST the many questions that have been discussed in our columns without arriving at any conclusive and thoroughly satisfactory answer are those relating to the hardening and tempering of steel, the expansion of iron with increase of temperature, the changes it undergoes between the molten and solid state, and the phenomenon of a piece of solid iron floating on a bath of molten and necessarily less dense metal. Lately, some elaborate and instructive experiments have been made by Mr. Thomas Wrightson, of Stockton-on-Tees, with the view of determining—(1) the changes in wrought and cast-iron when subjected to repeated heatings and coolings; (2) the effect upon bars and rings when different parts are cooled at different rates; and (3) the changes occurring in molten iron when passing from the solid to the liquid state, and *vice versa*. The results were embodied in a paper read before the Iron and Steel Institute, which, with the tables and diagrams accompanying it, will take the first place in the literature of the subject. It contains, in fact, the only really scientific attempt to settle some of the moot points that, so far as we know, has been published, and it is to be hoped that it will be issued in a separate form after it has appeared in the Transactions of the Institute. The reason that so little has been done hitherto in the accurate observation of the physical properties of iron is two-fold:—first, the molecular changes of the metals is so slow at ordinary temperatures, and under ordinary conditions of strain, that trustworthy observations, which must of necessity have extended over long periods, are difficult to obtain; secondly, when the temperatures are high—at which times the greatest and most rapid molecular changes are occurring—the difficulties of observation are increased to such an extent that the results lack the scientific accuracy which characterizes the chemical methods of examining iron. Hence, probably, chemical analysis has been employed almost to the exclusion of physical methods. However, to take the first heading, the changes effected by repeated heating and cooling, we have only to point to a steam-boiler to see how important it is that we should understand these changes not only as they affect iron, using the term in its generic sense, but also as they affect the different qualities and

various makes. In the case cited we have one side of the plates subjected to a fierce heat, while the other is at a comparatively moderate temperature approximating to that of the steam and water. Where a riveted seam occurs the conducting surfaces of the metal are thickened, and the results are not unfrequently seen in what are known as seam rips. Mr. Wrightson cut some strips from the plates of two long egg-ended boilers which had exploded by ripping at the seams, and found the iron had become brittle, was apparently crystalline in fracture, and had but small tensile strength. Strips taken further from the seam showed that in both cases the iron had been less injuriously affected. Chains are rendered brittle by the strains to which they are alternately subjected, and they are periodically annealed to avoid any danger from the change in the tenacity of the iron. Mr. Wrightson found that a similar treatment of the strips from the boiler-plates restored the fibrous character of the iron, as well as its ductility and tensile strength. The facts mentioned afford an excellent example of the importance of the physical effects produced by repeated changes of temperature. The change effected by one heating and cooling is so small that a cumulative test is the only method by which a really useful result can be obtained. If a wrought-iron bar is heated expansion takes place; if the bar is then suddenly cooled in water it contracts, and the amount of the contraction exceeds that of the previous expansion, so that the bar is now smaller than it was before being heated. If these operations are repeated, the contraction increases during many successive operations. Thus a bar $1\frac{1}{8}$ in. square by 30.05 in. long, heated to a dull red and cooled in water, contracted after first cooling .04 in.—after 15th cooling .68 in. or a percentage of 2.26 in fifteen operations. That was common iron; the best gave the same degree of contraction for the first two or three operations, but on the 15th cooling was found to have contracted only .56 in. or 1.86 per cent. of its length. Similar bars heated to redness and allowed to cool in air do not exhibit any change in dimensions, but if they are raised to a white heat a slight contraction does occur. Some wrought-iron hoops, welded up from bars of the same section as the iron employed in the previous experiments yielded, almost identical results—the contraction progressing steadily during twentyfive coolings. Pieces of wrought plate, planed up into a nearly accurate rectangular form, were then tested, with the result that after cooling in water a reduction of specfic gravity was discovered amounting to 1 per cent. after 50 coolings, and 1.57 per cent. after 100, further heatings and coolings not appearing to effect any change. A reduction of the surface takes place after each heating and cooling, due to two causes—viz., scaling, which amounts to .00057 in. reduction of thickness after each immersion in water; and a persistent contraction, which takes place vigorously after each immersion up to fifty, and probably to a much greater number. Bulging is noticed in the case of plates, which thicken towards the center, while the edges are thinned. Professor Stokes has explained the bulgings, the scaling is well understood, but the contraction can be accounted for only on the supposition that a molecular change in the iron occurs when the red-hot metal is suddenly cooled. Cast-iron bars, on the contrary, whether cooled suddenly in water or gradually in air, expanded slightly, while cast-copper rods exhibit no change when cooled in air, but expand slightly after being several times suddenly cooled in water. Some of the most valuable and interesting of Mr. Wrightson's experiments were made upon bars and rings partially immersed only when being cooled. From the results recorded above it would be imagined that if a thick hoop, or portion of a hoop, were cooled by partial immersion, the portion in the water would contract, while that allowed to cool in the air would exhibit no change. Such a conclusion, however, is entirely erroneous, for when a hoop 18 in. in diameter, forged out of iron $3\frac{1}{2}$ in. broad by $\frac{1}{2}$ in. thick, was half immersed in water, it showed, after twenty successive heatings and coolings, that the water-cooled edge had increased 1-24 in., or 2.14 per cent. in length, while the air-cooled half had contracted 7.9 in., or 13.65 per cent., so that the hoop had become a section of a cone. To ascertain whether the form

had anything to do with this curious result, a wrought-iron bar 3½ in. deep, by ½ in. thick, by 28.4 in. long, was treated in a similar manner, when the dipped edge was found to elongate, while the upper edge, cooled in air, contracted —so much so that the originally straight bar became curved, the water-cooled edge becoming convex, while that in the air became concave, After the twelfth immersion the lower edge was found to have expanded 44 in., while the upper had contracted 1.96 in. An experiment was also made to test the effect of reversing the bar. A similar bar to that previously described after five coolings was curved so that the versed sine of its air-cooled edge measured 1½ in. The concave edge was then placed in the water, and five immersions brought the bar within an ⅛ in. of the straight, and the eleventh cooling threw the concavity on the other side of the bar. These experiments were subsequently repeated in a more elaborate manner with steel and wrought-iron hoops accurately turned and bored, and immersed to varying depths. The explanation was furnished by Col. Clark, R.E., some years ago, after similar experiments made at Woolwich. The immersed portion, being quickest cooled, is the stronger metal, and tends to pull in the upper part, which, besides, contracts as it slowly cools, so that the excess of contraction rests with the air-cooled portion. Col. Clark does not appear to have noticed the expansion of the water-cooled edge. These experiments are unquestionably of value, and the knowledge gained by them may be utilized in the arts; hence it would be advisable to issue the tables and the diagrams, together with the descriptive matter, in a separate form.

In making the experiments above noticed Mr. Wrightson's attention was naturally drawn to the changes occurring in cast-iron when passing from the solid to the molten condition, and *vice versa*. It is well known that a piece of cold cast-iron will float upon a bath of molten cast-iron, and much has been written to account for the apparent anomaly that cast-iron which has, in cooling, contracted about 1 per cent. from the original size of the mould in which it was cast, should, with a presumably higher specific gravity than the molten metal, float in the same. The ordinary way in which the experiment is performed is to take a small piece of cold iron, and throw it into a ladle of molten metal. It sinks at first, and then rises buoyantly to the surface. The assumption is made that the sinking is due only to the momentum acquired in falling; but the fact is, the cold iron sinks because it is heavier, and rises to the surface as it becomes heated and expands—the density being thus diminished. Mr. Wrightson had spheres cast 1 in., 2 in., 3 in., 4 in., and 5 in. in diameter, which were lowered, not thrown, into the molten metal, and he devised an instrument which registered the amount of sinking and of flotation, below and above a line of equilibrium. The diagrams obtained with this instrument show that in every case the ball sinks when first lowered into the metal, the duration of the sinking period being, as a rule, longest with the smaller balls. The outside surface of a ball expanding only to a small degree affects the volume more in proportion as its diameter is larger (the surface varying as the square and the volume as the cube of the diameter.) Hence, as a rule, five or six seconds may be sufficient to expand a five-inch ball to the floating point, while a two-inch ball would require a much longer time before it would float. Thus the index, at the moment the ball enters the metal, crosses the line of equilibrium and sinks below it; then it rises gradually until it attains its maximum above the line, and then falls again as the ball melts until it reaches the line, the fork and spring holding the ball being adjusted to remain in equilibrium. It may be reasonably assumed that the diagrams read backwards convey a correct impression of the changes going on when molten iron is cooling to the solid. The mould is usually made about 1 per cent. larger than the casting is required to be, and the metal when poured in is always supposed to expand; hence the sharpness of the impressions. This expansion is marked on the diagrams by the quick rise of the line between the molten point and the point of greatest volume. Mr. Wrightson says that in carefully observing the cooling of a casting it will be noticed a considerable contraction takes place some minutes after the iron

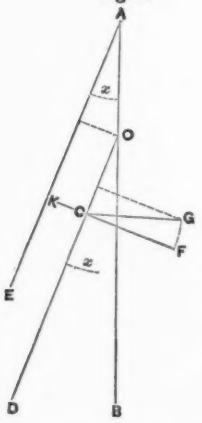
has partially set. A few minutes after the "git" is closed and congealed there is a sudden breaking open of the same, and a sponging out of the hot liquid metal from the interior, as though the internal part had suffered a sudden squeeze. This phenomenon is indicated by the sudden fall of the line on the diagram, and indicates in this case the passage from the plastic to the solid state. This explanation agrees with that which has been urged before by some of our correspondents, and has the advantage of almost a demonstration to recommend it. The instrument devised by Mr. Wrightson for use in these experiments will possibly be found of value in testing the specific gravity of samples of metal; but as those interested in these subjects are sure to obtain a copy of this paper we have done all that is necessary in calling attention to it here.

THE ICE BOAT; ITS POSSIBLE VELOCITY.

By Brvt. Maj.-Gen. Z. B. TOWER, Corps of Engineers, U. S. A.

Let A B be the direction of boat's motion.
O D projection of sail regarded as a plane surface.
A E second position of sail, after an interval of one second or unit of time.
F C direction of wind at right angles to sail.
G C another direction of wind.
x angle of sail with boat's direction.
w=F C=velocity of wind.
v=A O=velocity of boat.

Fig. I.



The pressure of wind upon sail, which gives motion to the boat, will be due to the velocity of the wind at right angles to the sail, less the motion of the sail in the same direction. Or to

$$\text{F C}-\text{CK}=w-v\sin x$$

This pressure, therefore, on a unit of surface is,

$$(w-v\sin x)^2\times \text{C}$$

C being a constant, which has been determined by experiment.

Pressure upon sail, whose surface is represented by S, will therefore be

$$\text{C. S.}(w-v\sin x)^2.$$

The component of this pressure or force in the direction of the boat's motion is

$$\text{C. S. }(w-v\sin x)^2\sin x.$$

Any other direction of the wind, as G C, gives component perpendicular to the sail, less than F C or w.

If sail be increased inversely with sine of angle of incidence of wind, resulting pressure on sail will be constant. But in that case, weight of moving mass, and resistance of air to boat's motion, will each be increased. Hence wind's direction at right angles to boat should give maximum velocity.

The propelling force of the wind will, when the boat has attained uniform motion, be equal to all resistances to that motion. If we represent those resistances by R, the equation of uniform motion becomes,

(1) $$\text{C. S. }(w-v\sin x)^2\sin x=\text{R}$$

The velocity of the boat will increase as resistance diminishes.

With resistance O, the resulting equation of motion becomes,

$$\text{C. S. } (w - v \sin x)^2 \sin x = 0$$

This equation will be satisfied by making $w - v \sin x = 0$.

If $\sin x = 1$, x becomes 90° and $v = w$, In other words, the sail being at right angles to the boat, there being no resistance, boat's velocity is equal to that of the wind, as it should be. In same equation $w - v \sin x = 0$, when $\sin x$ becomes 0, v becomes infinite. Hence, under the supposition made, the limits of the velocity v, are w, and infinity; the velocity v increasing, as $\sin x$ diminishes. The condition of no resistance cannot practically exist, but the discussion shows that, as the resistance diminishes, $\sin x$ diminishes and v increases.

The maximum of expression

$$\text{C. S. } (w - v \sin x)^2 \sin x = \text{C. S. } w^2 (1 - y \sin x)^2 \sin x$$

gives,

$$y = \frac{1}{3 \sin x},$$

in which y is a decreasing function of x.

The above equation, as well as an examination of Fig. 1, shows that great velocities are possible if resistances can be made small.

The resistances to boat's motion are due to friction of the runners on the ice, to the work of cutting or breaking the ice, and the resistance due to the wind, striking against parts of the boat so as to give a pressure opposed to its motion.

Friction is proportional to pressure, and is usually regarded as independent of the velocity of motion, unless the pressure is sufficiently great to change the form of either of the materials in contact. In this case the ice is cut and broken, and work is done. The amount of this work will depend upon the pressure, and the velocity with which it moves, possibly varying with squares of velocity.

Resistances due to the air will vary with the square of the velocity with which it strikes against surfaces of the boat. All the cross-pieces will tend to impede motion of boats as usually made, but its construction can be such as to reduce this resistance very essentially. Friction of the air against sail, and some surfaces of the boat impede its motion. This force, however, is small.

There are no experiments, that I am aware of, determining the value of co-efficient of friction in the sailing of ice boats.

The pressure producing friction is, first, the weight of the boat, and second, the component of wind's pressure upon the sail at right angles to the boat. The first will be represented by W, the second by $\text{C. S. } (w - v \sin x)^2 \cos x$.

Resultant pressure of these two forces will be

$$(3) \quad P' = \sqrt{W^2 + \text{C. S. } (w - v \sin x)^2 \cos x}^2$$

Put $\text{C. S. } (w - v \sin x)^2 \cos x = P$.

It is some proportional part of W, which can be expressed by variable c. Hence $Pc = W$, and by substitution in above formula

$$P' = \sqrt{1 + c^2}.\ P,$$

and the friction, f being its co-efficient, is

$$f \sqrt{1 + c^2}.\ P.$$

Air's resistance is expressed in pounds, and for each case will be some proportionate part of $\sqrt{1 + c^2}.\ P$. This function is variable, but will be determined hereafter. For the present it may be introduced as

$$f' \sqrt{1 + c^2}.\ P,$$

giving

$$R = (f + f') \sqrt{1 + c^2}.P$$

or, by putting f instead of $(f + f')$

$$f \sqrt{1 + c^2}.P = R$$

the total resistances to boat's motion. Hence,

$$\text{C. S. } (w - v \sin x)^2 \sin x = f \sqrt{1 + c^2}.\ P$$

$$\text{C. S. } (w - v \sin x)^2 \sin x =$$

$$f \sqrt{1 + c^2}. \times \text{C. S. } (w - v \sin x)^2 \cos x.$$

$$\tan x = f \sqrt{1 + c^2}$$

Putting for P its value $\frac{W}{c}$ there results

$$(4) \quad \text{C. S. } (w - v \sin x)^2 \sin x = \tan x \times \frac{W}{c}$$

Let $yw = v$, then $y = \frac{v}{w}$ is the ratio of boat's velocity of motion to that of wind; and (eq. 4) becomes

$$(4) \quad \text{C. S. } w^2 (1 - y \sin x)^2 \sin x = \tan x \times \frac{W}{c}$$

$$\tan x = f \sqrt{1 + c^2}.$$

These two equations are the equations of motion when the wind is at right angles to the sail, with f as an indeterminate variable. This variable f will be discussed further on.

As the weight of the boat must counterbalance the pressure of the wind on the sail, c can never be less than the ratio of the height of the center of figure of the sail to the lever arm of boat's weight. The form of the sail and its dimensions, with weight and size of boat, and distribution of its load, will determine this minimum limit of c.

The value of c corresponding to maximum value of y in the above equation (4) can be found by trial. A new equation, however, showing the relation of c to the constants and angles when y is a maximum, can be obtained by deducing value of y in equation (4), which is

$$y=\frac{1}{\sin x}-\frac{1}{\sqrt{\sin x}^3}\sqrt[3]{\frac{f\sqrt{1+c^2}}{c}}\times\sqrt[3]{\frac{W}{CSw^2}}$$

Differentiating and putting differential coefficient=O, the resulting equation, after a series of reductions, becomes

$$\text{(A.)}\quad \sqrt[3]{\frac{W}{CSw^2}}=\left\{\frac{3}{2\sqrt{c.\cos x}}+\frac{(1+c^2)}{2\sqrt{c\cos x}^3}\right.$$
$$\left.-\frac{1}{2.\sqrt{c}.\sqrt{\cos x}^3}\right.$$

The value of c that will satisfy this equation will correspond to y a maximum in eq. (4.)

Having fixed the values of the constants W. C. S. w^2, and assuming f, c can be found with a close approximation, by the process of successive substitutions.

The pressure of wind, with velocity of one mile an hour on a square foot of surface, is recorded as .005 lbs. On large surfaces it is very much increased, the ratio of increase being set down as large as 1.7. For sails containing 200 and 300 square feet, this unit of pressure will be taken as .008=C.

Friction of iron upon ice is doubtless small, but the work of cutting the ice, and the resistance of the wind, will increase rapidly with increased velocity of the boat.

The equations to be solved for a maximum value of y are

$$\text{C. S. } w^2\ (1-y\sin x)^2\sin x=\tan x\frac{W}{c}$$

$$\tan x=f\frac{\sqrt{1+c^2}}{c}$$

With sail's surface S=200 sq. ft. W=600 lbs. there results for

$f=\frac{1}{6}$, $w=25$ { maximum value of y, 1.265; corresponding value of c 1.9; angle, 19°.29′

with

$f=\frac{1}{6}$, $w=25$ { maximum value of y, is 1.052; corresponding value of c is 1.9; angle $x=$ 23°.14′

with sail's surface S=300 sq. ft. W=800 lbs.

$f=\frac{1}{4}$, $w=20$ { maximum value of y is 9.97; corresponding value of c is 2.3; angle $x=$ 22°.41′

with

$f=\frac{1}{6}$, $w=25$ { maximum value of y is 1.357; corresponding value of c is 1.8; angle $x=$ 18°.56′

with

$f=\frac{1}{6}$, $w=25$ { maximum value of y is 1.1414; corresponding value of c is 1.7; angle $x=$ 21°.32′

with

$f=\frac{1}{6}$, $w=30$ { maximum value of y is 1.454; corresponding value of c is 1.3; angle $x=$ 18°.10′

with

$f=\frac{1}{4}$, $w=30$ { maximum value of y is 1.285; corresponding value of c is 1.3; angle $x=$ 22°.04′

With sails surface S=350 sq. ft. W=800 lbs.

$f=\frac{1}{6}$, $w=30$ { maximum value of y is 1.1; corresponding value of c is 1.3; angle x 28°.40′

If f has not been assumed too small and C too large, the foregoing solutions indicate that an ice boat may attain a velocity much in excess of that of the wind that propels it.

From equation $R=\tan x\times\frac{W}{c}$ there results for $f=\frac{1}{4}$ (see preceding solutions).

R, the resistance to motion, $=\frac{315}{1000}W$,

while the velocity of the boat is 1.285 times that of the wind.

With $f=\frac{1}{8}$ the resistance becomes $\frac{42}{100}$W, while v is $\frac{11}{10}w$.

It seems improbable that the resistances to the motion of the boat can become so large a proportion of its weight, unless its construction be such as to expose large surfaces to the wind's resistance.

In equation of motion,

$$\text{C. S. } w^2(1-y\sin x)^2\sin x=\tan x\times\frac{\text{W}}{c},$$

by increasing S, the sail's surface, the propelling force will increase. Hence, if W remains constant, f may be increased proportionately with S, without reducing velocity v. Now, by elongation, the sail's surface can be increased to large dimensions, without increasing W, farther than by the additional weight of sail. One condition only is to be observed in this change, viz., that the ratio of the height of the sail's center of figure above the ice, to the lever arm of the boat's weight, must not exceed the value c.

The foregoing is intended simply as a discussion of the conditions essential to give an ice boat its maximum velocity, and also to show the probabilities of its maximum speed exceeding that of the wind. These conditions require that the wind shall be at right angles to the sail, which may be increased to large dimensions, so that the height be kept within the limits imposed by the value c. The boat itself should be so constructed as to present narrow surfaces to the wind, with a view to diminish the air's resistance.

GENERAL EQUATION.

Equation (4) was obtained under the condition that the direction of the wind should be at at right angles to the sail. If the wind make any angle, u, with the plane of the sail, the equation will be modified, and will apply to all cases. It will become

$$\text{C. S. } (w.\ \text{sin.}\ u-v\sin x)^2\sin x=\tan x.\frac{\text{W}}{c}.$$

If c were known, this equation could be solved for all assumed values of x and w. c, however, is a variable involving the variable f.

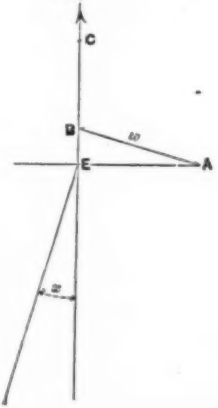
ANALYSIS OF THE VARIABLE f.

So far as f represents the co-efficient of friction proper, it may be represented by f a constant; and as the multiple of P′, expressing work done in cutting and breaking the ice and removing the fragments, by $f'v^d$. The resistance due to friction, therefore, will be $(f+f'v^d)$ P′.

To investigate the resistance of the air, let the boat's frame be trapezoidal in form, covered on its under surface with canvas, and on its upper with thin boards and canvas, with an iron frame-work to support the mast. The air's resistance will thus be due only to the boat's front end, the mast, the iron frame-work, very small, the side surfaces of two men, sitting at right angles to boat's length, and to surface of ballast.

The velocity with which the air strikes these surfaces at right angles to their planes, and directly opposed to boat's direction, will appear from Figs. (2 & 3).

Fig. 2.



In Fig. 2 let AB$=w$ be wind's direction at right angles to sail. EC$=v$, velocity of boat. Then

$$\text{CE}-\text{BE}=v-w\sin x,$$

will be the velocity with which the wind will strike the boat, in direction opposite to its motion.

The same result is found from an inspection of Fig. 3.

Since boat moves from a to b, while wind moves from a to d, the effect upon

Fig. 3.

the sail is the same as though the wind moved from b to a, with a velocity $b\,a$, the sail not moving, and that is in effect the direction in which the wind strikes the sail, and $b\,g = v - w \sin x$ is its effective velocity in direction $b\,a$, opposite to boat's motion.

If F represent the surfaces above enumerated, the air's resistance to motion will be,

$$\text{C. F. } (v - w \sin x)^2.$$

Make $\quad \text{C. F. } (v - w \sin x)^2 = e\text{W} = e\text{P}c.$

As e varies directly with the variable $(v - w \sin x)^2$, it will be equal to a constant multiplied $\times (v - w \sin x)^2$. Hence

$$\text{C. F. } (v - w \sin x)^2 = f''(v - w \sin x)^2 \text{P}c$$

$$f'' = \frac{\text{CF}}{\text{W}}$$

a known quantity, and

$$\text{tang.} x = (f + f' v^d)\sqrt{1 + c^2} + f'' c (v - w \sin x)^2$$

The general equations are

$$(5)\quad \begin{cases} \tan x = \left((f + f' v^d)\sqrt{1 + c^2} + f'' c\,(v + w \cos(u + x)\right) \\ \text{CS. } (w. \sin u - v \sin x)^2 = \tan x \dfrac{\text{W}}{c} \end{cases}$$

f, f' and d may be obtained by a series of careful trials.

After those constants are obtained, the two equations (5) will have but two unknown quantities, and can therefore be solved for any value of w and angle x.

The air rapidly passing obliquely behind the sail, doubtless diminishes the pressure upon its back surface, and thereby increases the effect of the winds propelling force. It virtually increases the constant C.

Underground Temperature.—The temperature of the surface of the ground is not sensibly influenced by the flow of heat upwards from below, but is determined by astronomical and atmospheric conditions. The rate of increase in traveling downwards from the surface may conveniently be called the *temperature gradient*, and averages about 1° F. for fifty or sixty feet. This is about five times as steep as the temperature gradient in the air.

If we draw isothermal surfaces for mean annual temperature in the ground their form beneath mountains and valleys will be flatter than that of the surface above them. This is true even of the uppermost; and the flattening increases as we pass to lower ones, until at a considerable depth they become sensibly horizontal planes. The temperature gradient is consequently steeper beneath gorges and least deep beneath ridges.

In a place where the surface of the ground and the isothermal surfaces beneath it are horizontal the flow of the heat will be vertical, and the same quantity of heat will flow across all sections which lie in the same vertical. In this case the flow across a horizontal area of unit size will be equal to the product of the *temperature gradient* by the *conductivity*, if we employ the latter term in an extended sense so as to make it include convection by the percolation of water, as well as conduction proper. It follows that in comparing different strata lying in the same vertical, the gradient will vary in the converse ratio of their conductivity. It seems probable that the same law of inverse proportion between gradient and conductivity holds approximately even when the strata

compared are not in the same vertical, but are widely distant.

As regards the modes of observation which have been employed for the determination of gradients:—shafts full of water, and wells of large diameter, afford so much facility for equalization of temperature, by currents between the colder water above and the warmer water below, that they furnish no useful results. Even in bores of small diameter the same disturbing cause exists and always makes the observed less than the true gradient.

Observations in mines will be vitiated by the presence of pyrites, which generates heat by its slow combustion, and are also liable to be vitiated by strong currents of air; but when they are taken at the newly exposed face of a gallery which is being driven into the rock, care being taken to prevent strong air-currents at the place, and the surrounding ground not being too much honeycombed by previous excavations, good results may be obtained. A hole should be bored to the depth of about two feet in the newly exposed face, the thermometer inserted, and the hole plugged with clay.

RAILWAY STATISTICS.—Among the interesting statistical details connected with railways which have been published in German, it appears that Germany itself holds the first place in Europe in the quantity of its railway communications, possessing a net wort of 30,364 kilometres Next comes England with 27,540 kilometres, then France with 23,883 kilometres, and Russia with 21,687 kilometres. Austro-Hungary has only 17,997 kilometres, and Italy 8213 kilometres. The railway mileage of the United States of America amounts to more than five-sixths of the total mileage of Europe, being 127,470 kilometres. The other States of America have altogether only 16,000 kilometres of railways. Asia has 14,000 kilometres, Australia 4000 kilometres, and Africa only 2900 kilometres.—*The Engineer.*

NEW PROCESS OF IRON SHIPBUILDING.—Mr. J. Humphreys, general manager of the Barrow Shipbuilding Company, has invented a new mode of constructing iron ships, on what is known as the longitudinal system, whereby the maximum strength is obtained with the minimum weight. The dependence on skilled manual labor, which is so eminently characteristic of the present method of shipbuilding, is nearly obviated, the frabric being most readily and simply constructed by the employment of channel sections of iron or steel, and the riveting effected by mechanical means such as Tweddell's. The fabric is erected in permanent buildings, specially designed for the purpose, whereby the present expensive systems of frame, floor-plate, bending, hand templating for the skin and other plating, shoeing and ribboning, hand riveting, &c., all of which are dependent on skilled manual labor, are done away with, mechanical appliances and unskilled labor taking their place, the result being that a stronger, lighter, and less expensive fabric is produced, and a considerable saving in the time required for building is effected.

REPORTS OF ENGINEERING SOCIETIES.

AMERICAN SOCIETY OF CIVIL ENGINEERS.—The latest published papers of the society are:

No. 186. The Use of Steel for Bridges, by Theodore Cooper.

No. 187. The Construction of the Atchison, Topeka and Santa Fe Railroad over the Raton Mountains, and the Performance of Locomotives on its Steep Grades, by James D. Burr.

INSTITUTION OF CIVIL ENGINEERS.—We have received, through the kindness of Mr. James Forrest, Secretary, the following excerpt minutes of the "Proceedings:"

Strength and Elasticity of Materials, by William James Millar.

The Traveling of Sea Beaches, by George Henry Kinahan, M. R. I. A.

The Street and Footwalk Pavements of Montreal, by Percival Walter St. George, A. I. C. E.

A Graphic Mode of Ascertaining the Flow of a Mill Stream, by William Shelford, M. I. C. E.

ENGINEERS' CLUB OF PHILADELPHIA.—At a regular meeting of the club, held October 18th, Mr. Rudolph Hering made some remarks upon the cement and works of the Copley Cement Co. The fact that their cement is not artificial, but depends entirely upon the nature and composition of the various layers of rock in quarry, leads to frequent variations in quality. Mr. Hering described processes of burning and grinding the rocks from which the celebrated brand, "Saylor's American Portland," is made. The rock is a massive argillaceous limestone, which occurs in beds of about twelve feet thick and forms the upper layers of the Trenton group.

Mr. Charles A. Ashburner read some notes on the "Eames' Petroleum-Puddling Process." This process for utilizing petroleum as a fuel has been in use for about five years, in the manufacture of blooms, boiler plate, horse-shoes, crucible and open hearth steel, and glassware. The petroleum vapor generator, which is the peculiar feature of this process, consists of a cast-iron vessel, with slightly-sloping shelves, projecting alternately from opposite sides. The petroleum enters at the top from a quarter-inch pipe and under twenty pounds pressure. Steam, heated to incandescence, enters the bottom of the generator, and meets the oil as it drips from the lowest shelf; they then pass into the combustion chamber, which is situated on the bridge-wall, and consists of a cellular layer of bricks placed on end and extending across the bridge. The gas is ignited here and forced into the furnace by an air-blast.

The works shut down at 6, p. m. At 6.15, a. m., the next day, fire is let into the furnace and puddling can be commenced at 7 o'clock. The capacity of a furnace is fifteen tons per day; to work which, ten barrels of crude oil are used. The iron manufactured by this process is of a very superior quality and commands a high price.

Mr. Rudolph Hering exhibited a full set of detailed drawings of the extensive works at present being prosecuted for the improvement of the sewerage of Boston.

Mr. Charles Darrach presented some notes on "Meyers' Formulæ for Proportionate Culverts on Railroads,"—these being in discussion of a paper by Mr. Cleeman, published in No. 3 of the Proceedings of the Club.

At the regular meeting of the club, held November 1st, Prof. L. M. Haupt read a report upon the "Progress of the Geodetic Survey in Pennsylvania," which was illustrated by maps and drawings. The report contains a synopsis of operations of the survey and results obtained. The inception of the survey was the result of a conversation held in 1874, between the Hon. C. G. Pattison, Superintendent U. S. Coast Survey, under whose direction the work is conducted, and Prof. J. P. Lesley, Director of the Second Geological Survey of Pennsylvania. Prof. Haupt was selected to conduct the work in this State, and began work by a reconnoissance in Lehigh and Northampton counties on July 1st, 1875. The manner of conducting the reconnoissance work was fully explained, and a list of points, as determined by the survey, given for reference.

IRON AND STEEL NOTES.

THE ART OF PUDDLING.—On the 7th October, Mr. R. Howson of Middlesbrough, read before the North Staffordshire Mining and Mechanical Engineers, a supplementary paper on "The Art of Puddling: its Past, Present, and Future." "What," he asked, "are the conditions which render the elimination of phosphorus a matter of such difficulty and uncertainty? To state the question in another way, What quality of pig iron metal, assuming it to be phosphoric, is best adapted for puddling purposes? The answer is, That which is most free from the silicious element. Remove the silicon from the pig, and the silica from the cinder, and the phosphorus will remove itself, provided there is enough carbon in it to keep it in a fluid state. It does not require a very critical knowledge of chemistry (which, indeed, I do not pretend to) to understand why this is the case.

"The silicon in pig iron has a higher affinity for oxygen than has phosphorus; *pro tanto*, therefore, it forestalls and prevents the oxidation of that element. Again, silicic acid has a higher affinity for the oxide base of the fettling than has phosphoric acid. It is even said that, under certain conditions, it will expel phosphoric acid already in combination with it. The presence of silica, consequently, in the cinder, tends in proportion to its amount to disturb the conditions by which alone phosphorus is oxidized, and its resulting acid is absorbed.

"If this view commends itself theoretically to the scientific chemist, as it undoubtedly does, then the corresponding practice ought to reveal its truth, and this is really the case. I have become convinced by experience that when the silicon in pig iron reaches 2 per cent. and upwards, while at the same time the phosphorus is high, say 1.5 per cent., it is a matter of extreme difficulty to obtain a good and invariable result as regards quality of puddled bar by one process only. On the other hand, when the silicious element falls below 1 per cent., then the presence of phosphorus presents but little difficulty, provided the fettling is sufficiently basic.

"I may here state that in the Cleveland district it so happens that while the phosphorus in the pig is very nearly constant (averaging 1.5 per cent.), the silicon varies in the most capricious manner, from 0.5 per cent, in exceptional cases, to as much as 5 per cent in others, It is this fact which has so often led to the complaints and strikes of the much-abused puddler, and it is this which has baffled all the attempts to reduce the art of puddling to a certainty. Moreover, it is this which has given such a marked advantage to the competition of steel made by the Bessemer and Siemens-Martin processes, in spite of their being (hitherto at least) only applicable to the more expensive brands of pig iron. By the latter methods, up to the present time, under all their modifications, the presence of phosphorus, except in very small quantity, is inadmissable. In puddling, on the contrary, neither phosphorus nor silicon presents any difficulty when taken alone. It is the combination of the two which offers such an unassailable front. Many members of this society will by this time have read the very important paper contributed by its former president, Mr. Adamson, to the Paris meeting of the Iron and Steel Institute. During the discussion on that paper he characterized the process of puddling as one of the manufacture of 'concrete.' With a germ of truth in it which cannot be denied, the term is, of course, an exaggeration. No one would more readily admit than Mr. Adamson that the process of puddling, provided only it is properly done, is cap-

able of producing wrought iron with properties which balance favorably with those of mild steel for a great variety of purposes. I will only refer to the well-known fact that iron rails made long ago, before the competition became so severe, were as a rule as good as the modern steel rail, with the property of durability, in many instances far exceeding that of steel. We have also the Staffordshire brands of bar iron, with their facility for welding, to which is due the reliabliity of such a manufacture as that of chain cables. With regard to plates, comparisons have been drawn in favor of steel on the ground that iron ones are deficient in lateral strength in consequence of their fibrous structure. Nevertheless there are not wanting facts tending to show that Yorkshire and Staffordshire plates will still hold their own against steel, especially with improved puddling, and a better system of rolling.

"The truth is, there is essentially very little difference between wrought iron and what is usually called mild steel. What small distinction does really exist is only such as to adapt each to its respective office. That which remains to be done in respect of wrought iron is to determine that we shall no longer adhere to a system of manufacture, the produce of which can, by any stretch of adverse criticism, be compared to concrete.

"I have called attention to the fact that when silicon is present in any notable quantity in pig iron, which is also phosphoric, it acts as a formidable preventive to the oxidation of the phosphorus. It has also been pointed out that an undue percentage of silicon in the fettling has a similar injurious effect. On the other hand, if silicon and its acid are comparatively absent, then there is but little difficulty with the phosphorus, and it is a corollary equally true, that if phosphorus is comparatively absent, then the presence of silicon is not detrimental. I will not trouble the meeting with a detail of my own experiments, but will rather confirm these statements by what I have been able lately to glean in France in connection with the working of the Godfrey and Howson furnace.

"At the Terre-Noire Company's works situated at Tamaris, they have had one of these machines at work for a considerable time, and a paper on the subject was to have been read at the last meeting of the Iron and Steel Institute had time permitted, by M. Escalle, the managing director. I have been favored by a perusal of this memoir, in which the author reports the removal of phosphorus to a greater extent than he has ever experienced before, while the operation of puddling with fluid iron taken direct from the blast furnace does not occupy a time of more than 25 minutes. On referring to the analysis of the pig iron, it was found to be low both in silicon and phosphorus, not much exceeding 0.5 per cent. of each constituent.

"At Hayanges, Messrs. De Wendel, after trying a series of experiments with this furnace, report a very favorable result. Their blast furnaces produce several qualities of pig metal, the extreme grey containing P 1.819 per cent. and Si 3.60 per cent., while the extreme white gives P 1.10 and Si 0.855, the carbon being nearly alike in both cases, viz., about 4 per cent. Observe the comparative deficiency of Si in the white metal. On inquiry, as might be anticipated, it was ascertained that it was this which was used, and, in fact, specially smelted for puddling purposes, while the grey was found to be entirely unsuitable.

"I may here note, by way of parenthesis, that in England grey iron is often employed when a superior quality of puddled bar is desired. The reason of this is that it contains more carbon, or carbon in such a condition as enables it to maintain the fluidity of the metal for a compartively long time. At Hayanges, it is evident that there is sufficient carbon in the white metal to effect the purpose, while its superiority for puddling consists in its low percentage of silicon.

"At the works of Messrs. Dupont & Fould, at Pompey, I witnessed the working of several heats. The pig iron contained on an average P 1.70 per cent., Si 1.4 per cent. The puddled bar was fairly good, but the P remaining in it was estimated to be not below 0.30 per cent. A more perfect elimination of this element could only be procured in this case by previously reducing the amount of silicon.

"At Fourchambault two days were spent, and a great number of heats witnessed, the duration of each heat averaging half an hour. In this instance the analysis of the pig showed a reversed condition to that of Hayanges. The P was low, 0.27 per cent., while the silicon was high, 3.0 per cent. If what I have stated above as to the effect of the interchange of these elements is correct, then the puddled produce of this pig ought to be good, and it proved to be so. The balls as they came from the furnace were hammered into solid homogeneous blooms, which were at once heated and rolled into finished bars direct and without piling.

"I could bring forward further evidence in illustration of the mutual counteraction of these troublesome constituents of pig iron, but will only refer in conclusion to the opinion of Mr. S. G. Thomas, a gentleman who has undoubtedly investigated the subject with great care and perseverance, and it may be added, with a certain amount of success in a quite unexpected direction. The whole of his theory is based upon the ascertained fact that an acid cinder prevents the elimination of phosphorus. If, therefore (he says in effect), we are to undertake the task of manufacturing steel out of phosphoric brands of pig iron, either by the Bessemer or the Siemens-Martin method, the lining of our converter or furnace must be basic to begin with, and the cinder arising from the metal during the process must be rendered basic. Whether this principle is applicable in practice at extreme high temperature or not remains still to be proved, but it is undoubtedly a true one in relation to the moderate temperatures of the puddling furnace.

"Leaving the hematites out of the question, we have in England a vast variety of iron ores more or less phosphoric. The utilization of these has of late years been gradually losing ground for want of the knowledge and applica-

tion of this simple fact—silicon and phosphorus are incompatible elements. Neither of them are formidable alone, but when both are present to any considerable extent, then our first study ought to be how to get rid of one of them. It seems certain at present, and likely to continue so, that in smelting phosphoric ores we must rest content with getting the whole, or nearly the whole, of the phosphorus in the pig metal. We have, therefore, only the silicon to deal with. How is it to be attacked? The rational way is to begin at the beginning, and smelt the iron specially for forge purposes, with a high burthen, a comparatively low temperature, and with sufficient lime to render the slag basic. What can be expected from the system of purchasing forge pigs hap-hazard in the open market, coming, it may be, off a foundry burthen, and containing an amount of phosphorus which may be easily surmised, but an absolutely unknown quantity of silicon? Care in puddling is all very well, but to make it successful care in the selection of brands is the first requisite. I believe it to be possible with any class of ores which we possess so to regulate the smelting process as to render mechanical puddling a comparatively simple operation. But it may be said that there will be always pig iron produced which, although of unsuitable character, must nevertheless be utilized. This no doubt is true, but if so, then it is all the more necessary to know what we have got before we commence to deal with it. To make good puddled bar out of pig metal, containing, as it often does, P 1.5 per cent. and Si 3.0 per cent., is a most difficult task, but if the greater part of the silicon is first removed, then it becomes an easy one. It thus happens that the old method of the refinery fire is founded on true principles, but it is too expensive, and must give way to more scientific modes of treatment. These are not wanting, although it would occupy too much time to discuss them at present. I can only conclude by repeating that it is the peculiar chemical antagonism which silicon and phosphorus (along with their oxygen compounds) hold to each other, that lies at the root of the difficulty of the manufacture of both wrought iron and steel out of the cheaper brands of pig iron. Simple as the fact may appear, it is by the proper appreciation of this law that we attain to success in mechanical puddling, and it is on this that we have to rely for the cheap production of steel in the Bessemer converter, as well as in the Siemens or Ponsard furnace."

RAILWAY NOTES.

REFERRING to the reforms on the Prussian State railways, the *New Free Press* says the Prussian Minister of Commerce has organized measures by means of which great economy will be effected. The principal changes are: (1) That steel rails shall be exclusively used. (2) All tunnels on new railways will be supplied only with a single line. (3) The optical signals between station and station will be given up, which plan has already been adopted with success in the United States and in certain European countries. This alteration does not involve the slightest disadvantage; on the contrary, it will get rid of a whole army of railway servants, who simply have to station themselves along the lines to signal the trains. 4) The introduction of a central position for sliding rail (switches) and signal apparatus, which can be set in motion by one man, and so a large number of switchmen at the small stations will be dispensed with. (5) The introduction of continuous brakes. These have been in vse since 1878 on nearly all passenger trains throughout European railways. (6) Relieving under certain conditions local lines, over which only light traffic passes, from some of the costly precautionary measures which are only absolutely necessary where the traffic consists of heavy loads or express trains.

THE St. Louis *Republican* gives the following estimate of the running expenses of a narrow gauge railroad, based on the performance of nineteen locomotives during the month of June last. The locomotives consume one ton of coal per seventy miles, one pint of oil for thirty-eight miles, one pound of tallow for seventy-seven miles of running. Engine repairs have cost $4\frac{3}{10}$ cents; the wages of engineers, firemen, and round-house men have cost $5\frac{9}{10}$ cents; fuel has cost $1\frac{1}{2}$ cents; and oil, tallow and waste have cost $\frac{1}{4}$ of a cent per mile run by the engines, making a total for engine services of 12 cents a mile; a result which is seldom equaled in the direction of economy. The gauge is we believe 3ft.

THE elevation above the level of the sea of some of the chief railways of the world out of the British Islands is as follows: The Apennine line at its highest point is 617 meters above the sea level; the Black Forest line, 850 meters; the Semmering, 890; and the railway over the Caucasus, 975 meters. The tunnel of the St. Gothard line attains an elevation of 1,154 meters; of the Bremner line, 1,367; and the Mont Cenis line, 1,338. In America the highest lines are the North Pacific, which at its most elevated point is 1,652 meters above the level of the ocean; the Central Pacific, 2,140; and the Union Pacific, 2,513 meters. The highest point of the railway over the Andes is 4,769 meters above the sea level.

IT is stated that the electric light is already in use on the Crown Prince Rudolph Railway in Austria as a headlight. The apparatus used consists of what is known as Schukert's dynamo-electric machine, a small three-cylinder steam engine and an electric lamp. It takes very little room on the locomotive, and is said to answer the purpose perfectly. On another Austrian road, the Emperor Ferdinand Northern, the electric light is carried on a car, or even removed from the car and set up at a station or other place—as the scene of an accident, or place of taking on or discharging troops in campaigns where there is no station, &c. The apparatus is the invention of Marcus and Eggar, of Vienna.

ENGINEERING STRUCTURES.

FLOODING THE SAHARA.—The French scheme of turning part of the Algerian Sahara into an inland sea continues to attract considerable attention in France, and scarcely a week passes without some allusion being made to it in the Paris Academy. At a recent sitting M. de Lesseps read a letter from Capt. Roudaire, in which the latter gave some details of the results of his sounding of the soil at various points, sands and marls being the beds most commonly met with. At one place, four meters below the surface, plenty of potable water was met with, which will be a great saving in carrying on the work.

At the same sitting MM. Ch. Martens and Ed. Desor presented several considerations against carrying out the plan, their opposition to it being shared by several other French men of science. They have themselves examined part of the ground which it is proposed to put under water, so that their opinions ought to have some weight. While giving every credit to M. Roudaire for the accuracy of the survey which he is carrying out, they, however, point out the difficulty of perfect acuracy, which in this case is all important, in the classic country of mirage, where the surface of the ground is constantly altered and deformed by reflection snd refraction. Moreover, they point out that to the south of the projected sea is the Wed-Souf, where are ripened the dates known as Tunis dates, the culture of which is a very special one. The least error in surveying, it is shown, might lead to the destruction of this culture, by allowing the waters of the Mediterranean to penetrate the soil where the date trees are grown, and thus destroy them. The authors do not attempt to touch the argument that even in historical times part of the Sahara now being surveyed was really a great lake; but they point out that there are proofs that in prehistoric times there must have existed an interior sea, at an epoch when the hydrographical conditions of Europe were very different from what they are now. In 1863, when exploring the region betwéen the oases of Guemar and the south extremity of the Shott Mebrir, they found the gypsum beds of the plateaux ended in regular lines like sedimentary beds, and from the soil they collected the *debris* of shells truly marine, such as *Buccinum giberrulum*, Lam., and *Balanus miser*, L. Above these shells, in the sand, they found *Cardium edule*, better preserved than they had ever seen it. Thus they found fossils characteristic of salt water, and of those which are a mixture of salt and sweet. The retirement of the waters from the Sahara the authors attribute to the elevation of the land, which is even yet below the level of the Mediterranean, and is to a great extent a network of salt lagoons.

It has been said that the creation of an interior sea, of 13,280 square kilometers, would change the pluviometric condition of the country, and even that of the whole of Algeria. This MM. Martens and Desor regard as a great illusion. Although the laws of the general atmospheric movements are little known, yet it is admitted that the Atlantic is the great reservoir from which come the vapors which are resolved into rain over the European continent, They believe that this is also the case for Africa. The Mediterranean is really only a gulf of the Atlantic, and they do not believe that an addition of 13,000 kilometers will add anything to its climatic influence. Long calculations have been made as to the quantity of water that would be evaporated by the new sea; but the authors point out that the predominating wind in the region is north, and that if it were rendered either too cold or too moist it would injuriously affect the date-culture carried on in the south. The surroundings of interior seas, like the Caspian and Aral, are steppes noted for their aridity; the shores of the Mediterranean suffer in the same way when, as last year, the rains of the north do not extend to the south. For these reasons MM. Martens and Desor think it would be a mistake to insist on the creation of the interior Saharan Sea.

In a subsequent sitting, however, it should be said, M. Favé endeavored to show that their fears were groundless, especially with regard to the accuracy of the survey; he thinks that the work in connection with the Suez Canal showed that perfect confidence may be placed in the methods of surveying adopted.

ORDNANCE AND NAVAL.

THE *Bureau Veritas* gives the number of merchant vessels of all countries as 54,921; of which 5,897 are steamers. The gross tonnage is 20,283,540, of which steamers have 6,173,935. England has 18,357 sailing ships, heading the list, and is followed by America, Norway, Germany, Italy, France, Greece, Sweden, Russia, Spain, Holland, Denmark, and other countries having less than 1000 each. As to steamers, England has 3,742; America, 519; France, 292; Germany, 244; Spain, 214; Sweden, 194; Russia, 156; Norway, 135; Holland, 113; Denmark, 101; Italy, 101, and other states less than 100 each.

THE Gallia, the City of Berlin, and the Germanic are all splendid types of merchant steamers. The Gallia is the latest addition to the Cunard fleet, which now consists of 46 vessels, having an aggregate of 84,354 tons of gross tonnage, and 12,988 nominal horse-power. She was built in 1878, by Messrs. James and George Thompson, of Glasgow. Her gross tonnage is 4,808 tons; and she is 5,300 horse-power indicated, and 700 nominal. She accommodates 300 cabin and 1,200 steerage passengers. Her principal dimensions are as follow: Length of keel, 430 feet, and over all, 450 feet; breadth of beam and depth of hold (both moulded), 44 feet and 33 feet respectively. She is an awning-decked or four-decked and bark-rigged ship, and both her upper or promenade deck and her spar deck are of iron throughout, whilst the main deck is of iron for two thirds of its length amidships. Throughout the engine and boiler compartment up to the lower

deck stringer, are deep web frames connected by brackets. The engine seat is formed by carrying the floors of the ship up solid to the bottom of the sole plate. It is then plated from one side of the ship to the other with ⅞th-inch plates. Her main coal bunkers, which are well-ventilated, afford storage room for 1,200 tons of coal, and there is an extra bunker forward, capable of holding 500 tons more, or it can be used for cargo if required. There are nine iron bulkheads throughout the ship, and seven of them run up to the spar deck, and are watertight. The general construction of the Gallia is such that she fulfills all the requirements of the Admiralty for employment in war or transport service She is fitted with three distinct systems of steering gear; first, there is Muir and Caldwell's patent steam-steering engine, which is operated from a wheel in a small house placed forward of the funnel and on the midship deck-house. Next there is a powerful hand-steering gear, provided with double wheels, which is placed in a wheel-house in the after part of the vessel; whilst in the third place there are arrangements for the use of auxiliary gear, in which purchase-blocks are employed. The engines of the Gallia are of the compound high and low pressure, direct acting, and three-cylinder type. The high pressure cylinder is 63 inches in diameter, whilst the two low pressure cylinders are each 80 inches in diameter, with a 5 feet stroke. Steam is supplied to the engines from eight cylindrical boilers 14 feet 6 inches in diameter and 9 feet 6 inches long, each boiler having three furnaces. The working pressure is 75 lbs. per square inch. There is an auxiliary boiler for working the donkey pumps and fire-engines. The propeller of the Gallia has an iron boss weighing about 10 tons, and fitted with movable steel blades, the propeller weighing in all about 24 tons. The speeds hitherto attained by the Gallia under circumstances of adverse weather have ranged up to 383 knots per 24 hours, and she has made the run from Queenstown to New York in 7 days 19 hours.

The City of Berlin was built by Messrs. Caird & Co., of Greenock, in 1875, and, next to the Great Eastern, is the largest merchant steamer afloat. She is 5,491 gross tons, and 3,139 tons net register. Her length is 489 feet keel and fore rake; depth, 36 feet; breadth, 45 feet. She has accommodation for 220 saloon and 1,200 third-class passengers. Her engines are 1,000 nominal horse-power, and 5,200 indicated. They are inverted direct-acting compounds, two cylinders, low pressure, having a diameter of 120 inches; high pressure, 72 inches; stroke, 5 feet 6 inches; steam, 75 lbs.; vacuum, 26 inches; revolutions, 56; expansion, 20 inches; speed, 16 knots. The City of Berlin has made the run from New York to Queenstown (2,830 knots) in 7 days, 14 hours, 2 minutes, being equal to nearly 15¾ knots per hour. She made in one day's running 405 knots, equal to 490 statute miles, or an average of 19 statute miles per hour.

The Germanic is the latest addition to the White Star line, and, like her sister vessels, is well known for the remarkable regularity of her speed. She has made the passage from Queenstown to New York (Sandy Hook) in 7 days, 11 hours, 37 minutes; and from New York (Sandy Hook) to Queenstown in 7 days, 15 hours, 17 minutes. The average speed per hour, winter and summer across the Atlantic, is 15½ knots, and the greatest day's run 410 knots, or 19½ statute miles per hour of 24 hours, consecutively. Her dimensions are: Length, 476 feet 6 inches; breadth, 45 feet 2 inches; depth, 33 feet 7 inches; nominal horse-power, 760; indicated, 5,431. The diameter of the two high pressure cylinders, each, is 48 inches: two low pressmre cylinders, each 83 inches. The length of stroke is 5 feet.

BOOK NOTICES.

ANATOMY FOR ARTISTS. By JOHN MARSHALL, F. R. S. New York: Macmillan & Co.

This work is of imperial octavo size of 430 pages, and illustrated by 200 wood-cuts interspersed in the text.

The anatomy relates to the human species, and it is so fully treated by text and figures, that it seems safe to affirm that, so far as bones and muscles are concerned, a more minute knowledge of structure and function is here presented for artists than is usually insisted upon in the case of physicians and surgeons.

THE FIELD ENGINEER; A HANDY BOOK OF PRACTICE IN THE SURVEY, LOCATION AND TRACK-WORK OF RAILROADS. By WM. FINDLAY SHUNK, C. E. New York: D. Van Nostrand.

This is designed to aid the young field engineer, whose technical training has been of limited amount, explanations of all field operations are very clearly given, and the theoretical discussions of the various cases are presented with exceptional fullness.

All branches of railway surveying are treated with the same reference to the needs of the inexperienced field worker. Section leveling, cross-section work, curve locations, with many methods of passing obstacles, either on tangent or on curves, are arranged in proper order, and illustrated by an abundance of cuts and by hypothetical cases.

Besides all the common tables of the engineers' table book, some useful, but unusual, ones are also added; one of chords, versed sines and external secants will prove especially acceptable to many engineers.

IRON BRIDGES AND ROOFS. By AUGUST RITTER. TRANSLATED FROM THE GERMAN By H. R. SANKEY, OF THE ROYAL ENGINEERS. London: E. & F. N. Spon.

Ritter's method has long been a favorite one with many engineers, and a familiar one to nearly all who have had occasion to employ different methods of analysis in the investigation of strains of bridges and roofs. The general method which is denominated Ritter's is easily acquired, and many students can apply it who have never read Ritter's treatise. The few, comparatively speaking, who had read the

German book, had diffused a knowledge of this elegant method so generally, that nearly all who cared to know it had acquired it.

It has thus happened that for a number of years the book has been in use in this country but only in the German.

Its English dress in which it now appears for the first time is quite an elegant one, the typography and cuts being excellent.

ELEMENTS OF THE DIFFERENTIAL CALCULUS, WITH EXAMPLES AND APPLICATIONS. By W. E. BYERLY, Ph.D. Boston; Ginn & Heath.

This is the work of a practical teacher, and is the outcome of his experiences in his professional work at Cornell and Harvard Universities.

The peculiarities of the work as set forth in the preface are, the vigorous use of the Doctrine of Limits, as a foundation of the subject, and as preliminary to the adoption of the more direct and practically convenient infinitesimal notation and nomenclature; the early introduction of a few simple formulas and methods for integrating; a rather elaborate treatment of the use of infinitesimals in pure geometry; and the introduction throughout of numerous applications to practical problems in geometry and mechanics.

NOTES ON RAILROAD ACCIDENTS. By CHARLES FRANCIS ADAMS, JR. New York: G. P. Putnam's Sons.

To learn how deeply interesting such an unpromising subject can be made, it is necessary to begin this book. Finishing it soon follows.

The reader is no less absorbed by the brief narrations of the book than by the philosophical comments upon the cause and cure of each particular class of accidents.

The book is in every sense an instructive one, and the author will have stimulated by this intensely interesting essay, inventive powers which neither disaster nor dry statistics could excite to action in the desired direction.

The concluding paragraph of the book will exhibit the style and spirit of the work:

"After all, as was said in the beginning of the present volume, it is not the danger but the safety of the railroad which should excite our special wonder. If any one doubts this, it is very easy to satisfy himself of the fact—that is, if by nature he is gifted with the slightest spark of imagination. It is but necessary to stand once on the platform of a way-station and to look at an express train dashing by. There are few sights finer; few better calculated to quicken the pulse. It is most striking at night. The glare of the headlight, the rush and throb of the locomotive—the connecting rod and driving wheels of which seem instinct with nervous life—the flashing lamps in the cars, and the final whirl of dust in which the red tail-lights vanish almost as soon as they are seen—all this is well calculated to excite our admiration; but the special and unending cause for wonder is how, in case of accident, anything whatever is left of the train. As it plunges into the darkness it would seem to be inevitable that something must happen, and that, whatever happens, it must necessarily involve both the train and every one in it in utter and irremediable destruction. Here is a body, weighing in the neighborhood of two hundred tons, moving over the face of the earth at a speed of sixty feet a second and held to its course only by two slender lines of iron rails; and yet it is safe! We have seen how when, half a century ago, the possibility of something remotely like this was first discussed, a writer in the *British Quarterly* earned for himself a lasting fame by using the expression that 'We should as soon expect people to suffer themselves to be fired off upon one of Congreve's *ricochet* rockets, as to trust themselves to the mercy of such a machine, going at such a rate;' while Lord Brougham exclaimed that 'the folly of seven hundred people going fifteen miles an hour, in six trains, exceeds belief.' At the time they wrote, the chances were ninety-nine in a hundred that both reviewer and correspondent were right; and yet, because reality, not for the first nor the last time, saw fit to outstrip the wildest flights of imagination, the former at least blundered, by being prudent, into an immortality of ridicule. The thing, however, is still none the less a miracle because it is with us a matter of daily observation. That, indeed, is the most miraculous part of it. At all hours of the day and of the night, during every season of the year, this movement is going on. It never wholly stops. It depends for its even action on every conceivable contingency, from the disciplined vigilance of thousands of employés to the condition of the atmosphere, the heat of an axle, or the strength of a nail. The vast machine is in constant motion, and the derangement of a single one of a myriad of conditions may at any moment occasion one of those inequalities of movement which are known as accidents. Yet at the end of the year, of the hundreds of millions of passengers fewer have lost their lives through these accidents than have been murdered in cold blood. Not without reason, therefore, has it been asserted that, viewing at once the speed, the certainty, and the safety with which the intricate movement of modern life is carried on, there is no more creditable monument to human care, human skill, and human foresight than the statistics of railroad accidents."

MISCELLANEOUS.

AN extensive district of tin ore is reported to have been found some fifty odd miles south-east from Sonora, in Mexico. The largest ledge will average about twelve feet wide, and has been traced half a mile. Assays made at Tucson by Professor Richard give 65.5 per cent. of tin. These ledges have been detected over a district of five miles, about thirty miles from the Gulf of California.

CHICAGO has a very efficient drainge system. It may prove useful to some of our readers to know that the following minimum inclinations were given to the sewers, which produce

a velocity of about 2ft. per second when running half full;—From 6ft. to 4ft., 1 in 2500; 3½ft., 1 in 2000: 3ft., 1 in 1666: 2½ft., 1 in 1250; 2ft., in 1000; 1¼ft. and 1ft., 1 in 500. In places where greater inclinations could be had, they were given.

ONE of the silk manufacturing firms of Lyons, France, are introducing the production of photographic impressions on stuffs. They sent to a recent meeting of the Photographic Society several pieces of silk with a variety of photographic pictures printed thereon, including, among others, a number of large medallions representing pictures of the old masters. The length of the specimens thus exhibited is stated as being no less than 13ft. The process by which they are produced is not given, but it is believed, says the *Commercial Bulletin*, that the prints are made with salts of silver. What will be the next possible thing in woven fabrics?

VERY numerous alloys resembling gold and silver have been patented. The last which have came to our knowledge are the following:—Gold alloy; 800 parts of copper, 28 of platinum, and 20 of tungestic acid are melted in a crucible under a flux, and the melted mass poured out into alkaline water, so as to granulate it. It is then melted together with 170 parts of gold. Silver alloy; 65 parts of iron and 4 parts of tungsten are melted together and granulated; also 23 parts of nickel, 5 of aluminum, and 5 of copper, in a separate crucible, to which is added a piece of sodium, in order to prevent oxidation. The two granulated alloys are then melted together. Both alloys resist the action of sulphuretted hydrogen.

THE United States now make about as much pig iron as Great Britain did in 1850. The increase in the make of pig iron in America from 1850 to 1878 has averaged 62,071 tons per year—equal to the production of two blast furnaces. At this rate of increase it would require over sixty-four years for the States to attain the output of Great Britain last year. American imports of rails from Great Britain are rapidly increasing, having been 7738 tons the first six months of this year, against 464 in the first half of last year and 2505 in the corresponding part of 1877. Of the 7738 tons—imported this year only 301 tons were iron rails, the remainder—7437 tons—being steel. This is exclusive of Vanderbilt's 12,000 tons which have not been shipped up to the end of June.

ADELAIDE PORT DOCK.—The directors of the Adelaide Port Dock Company have let a contract for the construction of a swing bridge at the dock entrance, to an American Iron Company. The general design of the structure is on the American pattern, showing great depth of truss, securely braced at top and bottom, and with the main ties and struts attached to the chords by means of steel pins. The bridge is to be entirely of iron, with the exception of the floor beams and roadway, which are to be of wood. The bridge is to swing on a center pivot at the south side of the dock entrance, with a long arm, 84 ft. in length, crossing the canal, and a short arm, 43 ft. long, and loaded with 60 tons of counterweight to balance. The turning machinery is to be on Seller's pattern, with a ring of conical steel rollers revolving on steel plates, and an outer circle of wheels for steadying purposes. These turntables are effective, and it is specified in this instance that two men working hand-levers shall be able to open the bridge easily, a pretty good test, considering that the weight of the structure will be close on 200 tons. The principal dimensions are: Length on centre line, 127 ft.; width, 30 ft.; height of truss above roadway, 16 ft. The bridge will be strong enough to carry a railway. The cost of the iron-work, including erection in place and painting, is 2650*l.*, and the total cost of the whole bridge, with foundation, is estimated at under 3500*l.* The engineer for the dock company is Mr. G. Chamier, C.E., and the contractor is Mr. Bates, for the Edge Moor Iron Company, Delaware, United States. The dock excavation is being proceeded with rapidly, and already considerably more than half the total amount of earthwork has been taken out. The land piles have been driven, and a large amount of the jarrah timber for the wharves has arrived, so that a start for the principal timber work is expected shortly. The most fortunate circumstance in connection with this contract has been the dryness of the excavation. Judging from trial holes, and from the indications at various pits, it was pretty generally supposed that the work of pumping the dock basin dry would be somewhat alarming. To Mr. Hickson, the South Australian Government Engineer of Harbors and Jetties, is due the credit of having predicted otherwise, for he reported that the ground would be excavated dry, although that opinion was not shared by most people. As a fact, the excavation has been surprisingly dry.

THE owners of the Great Eastern have, it is stated, at last determined to adopt a suggestion that has been frequently made, and are about to convert the great ship into a meat-carrying trader between London and Texas. The requisite alterations, which include new boilers, will involve an expenditure of about £100,000; but, as the vessel can carry 2000 head of cattle, or 36,000 sheep, the speculation should prove remunerative.

THE United States Consul in Birmingham has just made known that a circular has been issued by the Government of the United States of importance to shippers. The consul states that it is the intention of the Treasury Department to require the insertion in invoices of a clause showing that the invoiced prices include the charges for packing, shipping, &c., only when such is actually the case; and when the invoice does not properly include these charges, a specification in detail of the charges is to be given in the invoice in all cases.